Department of Physics, Princeton University

Graduate Preliminary Examination
Part I

Thursday, May 10, 2018
9:00 am - 12:00 noon

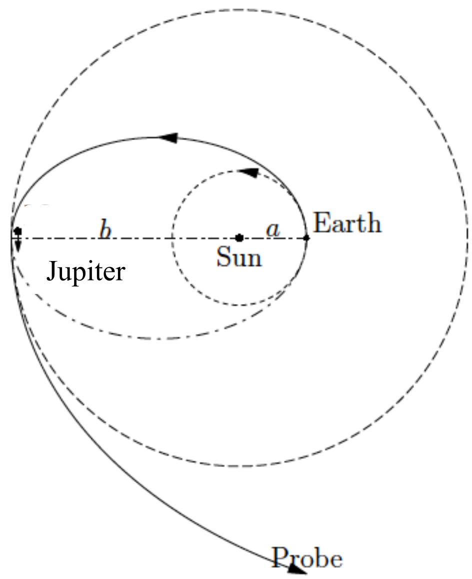
Answer TWO out of the THREE questions in Section A (Mechanics) and TWO out of the THREE questions in Section B (Electricity and Magnetism).

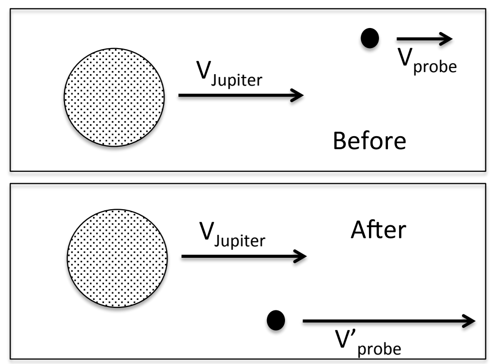
Work each problem in a separate examination booklet.

Be sure to label each booklet with your name, the section name, and the problem number.

Section A. Mechanics

1. **Planetary Slingshot**





A space probe is launched from Earth into a transfer orbit that takes it very near to Jupiter. As shown in the left figure, the transfer orbit is such that the probe reaches Jupiter's orbit just as Jupiter is arriving at the same point in its orbit, with the probe moving in the same direction as Jupiter. A blowup of the Jupiter-probe encounter is shown in the right figure. Things are arranged so that instead of the probe crashing into Jupiter's surface, it just misses and swings around the planet in a very tight orbit, emerging from the encounter moving in very nearly its original direction, but with a changed speed.

Make the approximations that the orbits of Earth and Jupiter are circular with radii a and b, respectively, and that the masses of Earth and Jupiter do not affect the transfer orbit between the two planets. Assume also that the masses of the Earth and Sun can be ignored during the near collision between the probe and Jupiter, and that the masses of Earth and Jupiter can again be ignored after the near collision.

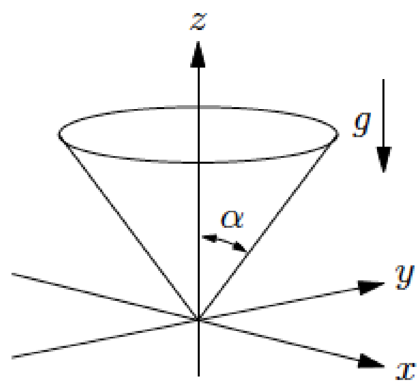
(a) Use conservation laws to evaluate the probe speed v_{probe} just before the rendezvous (v_{probe} in the frame of the Sun) in terms of Jupiter's orbital speed v_J and the radii a and b. Verify that Jupiter is moving faster than the probe.

(b) Treating the interaction between Jupiter and the probe as an elastic collision in one dimension, find the post-collision speed (relative to the Sun) of the probe, v'_{probe}. This is of course not accurate, since the hyperbolic orbit that just grazes the planet's surface will have a scattering angle less than π, but do make this simplifying approximation.

(c) Is it possible that the probe can then escape from the solar system? Find the condition that must be met in order for escape to occur.

2. **Particle in Cone**

A point particle of mass m is constrained to slide, without friction, on the inside of a circular cone whose vertex is at the origin and whose axis is along the z-axis. The half angle at the apex of the cone is α, as shown, and there is a uniform gravitational field $\vec{g}$, directed downward and parallel to the axis of the cone.



(a) Determine a set of generalized coordinates, and obtain the Lagrangian equations of motion in these coordinates. Identify any constants of the motion.

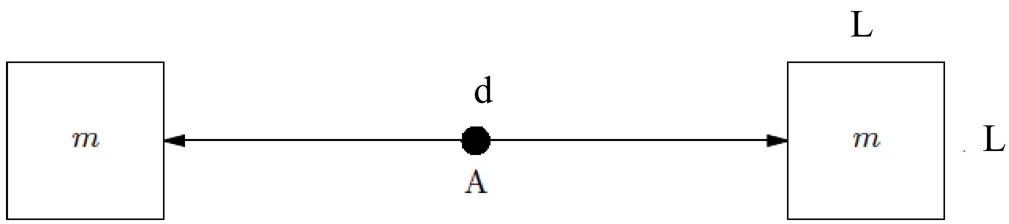
(b) Show that motion in a circular orbit at a fixed height z_0 is a solution of the equations of motion. Obtain an expression for the frequency ω of this orbit.

(c) Suppose that the particle moving in a circular orbit at height z_0 is given a small 'kick' in the direction away from the origin. Show that the subsequent motion consists of small oscillations about the unperturbed motion, and find the frequency Ω of those oscillations.

(d) Under what conditions will this perturbed motion trace out a single fixed orbit (as opposed to an orbit that either precesses or closes only after multiple revolutions)?

3. **Space Panels**

A body made of two rigidly linked panels is placed in outer space, where it can spin about 3 different axes. The panels are two thin squares of side L and uniform mass density, each with total mass m. The two panels are coplanar and are connected by a rigid beam (not shown in the figure) of negligible mass and length d (note that d is the edge-to-edge separation of the two panels, not the center-to-center distance).



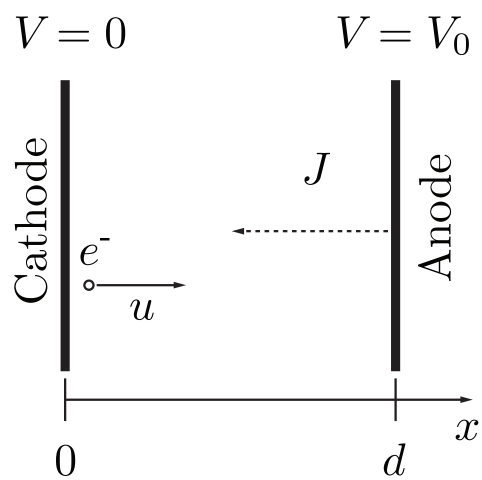
(a) Compute the principal moments of inertia $I_1 > I_2 > I_3$ for rotation about the center of mass point A. Indicate the direction of the three principal axes on the diagram

(b) After its construction, the set of panels was set spinning about the axis with the intermediate moment of inertia I_2, with its angular velocity chosen so that the pseudo-gravity at the center of each square section is $g/6$. Sadly, a tiny asteroid came by soon after and its impact nudged the angular velocity a little bit away from the I_2 axis. Show that the resulting motion of the panels will be perturbed strongly. What is the characteristic time for the growth of the perturbation?

Section B. Electricity and Magnetism

1. **Space charge**

   Consider two parallel plates of infinite extent separated by distance d. A constant potential difference $V_0 > 0$ is maintained between the cathode and the anode. Electrons are released from the cathode at zero potential with negligible velocity and are accelerated to the anode by the electric field. The region between the plates is a vacuum except for the electrons that are emitted into it. This leads to a finite space charge density, $\rho(x)$, where x is the distance away from the cathode (see picture). Under steady state conditions, $\rho(x)$ is independent of time, and the continuity equation implies that the current density $J = \rho u$ is independent of x, where $u(x)$ is the velocity of the electrons.

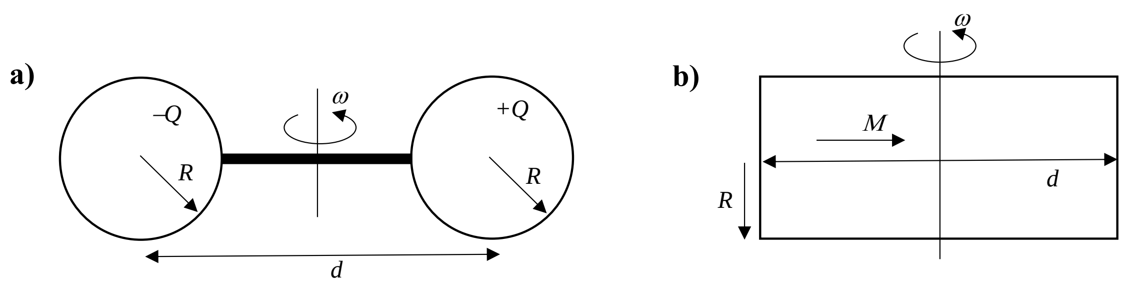


   a) Use Poisson's equation and conservation of energy to find the potential $V(x)$ as a function of x.

   b) Find an explicit expression for the current density J in terms of V_0.

2. **Mechanical Radio Station**

It is difficult and very inefficient to broadcast radio waves at very low frequencies. One of the recent ideas is to use a mechanically rotated electric or magnetic dipole, as illustrated in the figure below.



a) Consider a dumbbell consisting of two conductive spheres of radius R separated by a distance d. They have opposite electric charges $+Q$ and $-Q$. The spheres are mechanically connected by a thin insulating rod. You can assume $d >> R$ and ignore the mutual capacitance and the image charges on the spheres. Calculate the total power radiated by the dumbbell when it is rotated around the vertical axis passing through its center at a frequency ω.

b) Consider a permanent magnet in the shape of a cylinder of radius R and length d. The cylinder is uniformly magnetized parallel to its axis with magnetization M. Calculate the total power radiated by the permanent magnet when it is rotated around the vertical axis passing through its center at a frequency ω.

c) Assume the spheres are charged so the voltages on the spheres are $V = \pm 10$ kV relative to infinity. The permanent magnet is magnetized so the magnetic field at its center is equal to 1 T (assuming $d \gg R$). The radius R is both cases is equal to 0.1 m and the distance $d = 1$ m. The frequency of rotation is 10 kHz. Roughly estimate the power radiated by each source. Which of the sources will radiate more power?

3. **Dielectric Cylinder in an Electric Field**

An infinitely long cylinder of radius a and dielectric constant ϵ is placed in an initially uniform electric field of strength E_0. The axis of the cylinder is oriented at a right angle to the direction of the field.

a) Find the electric potential $\Phi(r; \theta; z)$ both inside and outside of the cylinder, in cylindrical coordinates $(r; \theta; z)$, where the z axis is the axis of the cylinder.

b) Find the electric fields $\mathbf{E}$ and $\mathbf{D}$ inside the cylinder.

c) What is the surface polarization (bound charge) density σ_b at $r = a$? What is the volume polarization charge density ρ_b for $r < a$?

d) What is the electrostatic energy per unit length inside the cylinder?

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Graduate Preliminary Examination
Part II

Friday, May 11, 2018
9:00 am - 12:00 noon

Answer TWO out of the THREE questions in Section A (Quantum Mechanics) and TWO out of the THREE questions in Section B (Thermodynamics and Statistical Mechanics).

Work each problem in a separate booklet. Be sure to label each booklet with your name, the section name, and the problem number.

Section A. Quantum Mechanics

1. **Hydrogen Molecule**

   Consider a hydrogen molecule, H_2, the bound state of two protons and two electrons.

   a) Write down the Hamiltonian, keeping only the kinetic energy terms and the Coulomb interactions of all the constituents, and omitting any terms which cause fine and hyperfine structure.

   In what follows, neglect all fine and hyperfine structure effects, and neglect the overall center of mass motion, but do include the spins and the relative motion of all the constituents:

   b) What is the degeneracy of the ground state(s)? Give all quantum numbers and symmetries of the ground state(s), including of the electron and proton spin degrees of freedom. Explain.

   c) What is the degeneracy, and what are all the quantum numbers of the first excited state(s) of this molecule? Explain.

   d) What is the energy difference between ground and first excited states? Estimate it first through a formula in terms of properties of the molecule's ground state, and then give an order of magnitude estimate in electron-Volts (eV). Explain.

2. **Flipping a spin**

A particle of spin one-half is at rest in a static magnetic field $\vec{B} = B_0\hat{z}$ oriented along the z-axis. The two-component wave function $\psi(t)$ of this system can be manipulated by turning on a magnetic field $\vec{B}_1(t) = B_1 \cos\omega t\,\hat{x} + B_1 \sin\omega t\,\hat{y}$ that rotates in the xy plane with a frequency ω. The time-dependent Hamiltonian that governs the evolution of $\psi(t)$ is

$$H = \mu B_0 \sigma_z + \mu B_1(\sigma_x \cos\omega t + \sigma_y \sin\omega t)$$

where σ_{xyz} are the Pauli matrices and μ is the particle's magnetic moment. We will discuss ways of choosing parameters so that quantum evolution over a time T transforms a $\sigma_z = -1$ state into a $\sigma_z = +1$ state.

(a) As a first step toward a solution, show that the 'interaction picture' wave function $\widehat{\psi} = \exp(i\,\omega t \sigma_z/2)\psi(t)$ evolves according to a time-independent Hamiltonian H_{rot}.

(b) Let the rotating field be turned on during the time interval $[0, T]$. Find ω and T such that a $\sigma_z = -1$ state at time $t = 0$ is perfectly transformed by this time evolution into the $\sigma_z = +1$ state at time $t = T$.

We remind you of a couple of algebraic facts that may be helpful in working through this problem:

- For any 2x2 matrices A and B: $e^A B e^{-A} = B + [A, B] + \frac{1}{2!}[A, [A, B]] + \ldots$
- For unit vector $\hat{n}$ and 2x2 unit matrix I_2: $\exp(i\Omega\hat{n}\cdot\vec{\sigma}) = \cos\Omega\, I_2 + i\sin\Omega\,\hat{n}\cdot\vec{\sigma}$

3. **Low-energy scattering**

Consider the problem of s-wave ($l = 0$) scattering of a particle of mass m from an attractive square-well potential of depth V_0 and radius r_0: $V(r) = -V_0\theta(r_0 - r)$ in three dimensions.

a) First consider the problem of s-wave bound states in this potential. Show that there is a critical potential strength V_{crit} such that for $0 < V_0 < V_{crit}$ there are no bound states. To put it another way, show that a bound state first appears when $V_0 = V_{crit}$. Determine the value of V_{crit}.

b) Now consider scattering of a particle of momentum k from this potential. Set up the equation for determining the phase shift $\delta_0(k)$ and show that it implies that $\delta_0 \sim A\, k$ as $k \to 0$. Evaluate the coefficient A as a function of V_0.

c) Calculate the contribution of the s-wave phase shift to the total cross section in the limit of small k. How does the zero-energy cross section behave: (i) in the limit $V_0 \to 0$?; (ii) in the limit $V_0 \to V_{crit}$? Comment and explain.

Section B. Statistical Mechanics and Thermodynamics

1. **Interstitials and Vacancies**

   Consider a crystal lattice where the atoms can sit on a lattice site or they can sit on an "interstitial" site. Assume the number of atoms N is equal to the number of lattice sites. The ground state has all atoms on lattice sites, so there are no interstitials and no vacancies (a vacancy is a lattice site that is empty; an interstitial is an atom that sits on an interstitial site).

   However, when the lattice is in thermal equilibrium at a nonzero temperature T, vacancies and interstitials are present at some nonzero equilibrium density. The number of interstitial *sites* is $N_i = \rho N$. The excess energy of an atom on an interstitial site is $\epsilon > 0$, and assume that there are otherwise no other interactions, so the crystal's energy is $E = K\epsilon$, where K is the number of interstitials. Each site can be occupied by at most one atom, so the number of vacancies is necessarily equal to the number of interstitials. Consider the thermodynamic limit of an infinite such crystal, with a given ρ and ϵ:

   a) For a given density of interstitials per lattice site $n = K/N$ (and the same density of vacancies), what is the total entropy per lattice site, $s(n)$?

   b) Calculate the equilibrium density of interstitials $n(T)$. Explain the leading behavior at low temperature T.

2. **Brownian Motion**

A solid spherical particle of radius b and mass M is suspended in a fluid, and is seen, using an optical microscope, to undergo Brownian motion with trajectory $\vec{r}(t)$. In this problem, you are asked to show that a measurement of the mean-square displacement, $\langle|\vec{r}(t_1) - \vec{r}(t_2)|^2\rangle$, can be used to determine Boltzmann's constant, k_B.

Assume the densities of the solid and fluid are identical, so buoyancy can be ignored. The cause of the Brownian motion is the rapidly fluctuating random force, $\vec{F}(t)$, due to collisions with the molecules of the fluid. The force has mean zero, $\langle\vec{F}(t)\rangle = 0$, and assume that its components (F_x, F_y, F_z) have two-time correlations of the form

$$\langle F_\alpha(t_1)F_\beta(t_2)\rangle = C\delta_{\alpha\beta}\ \delta(t_1 - t_2)$$

in terms of Kronecker and Dirac delta functions. The fluid has viscosity η and the system is isothermal at temperature T. Assume that the equation of motion of the particle is

$$M\frac{d^2\vec{r}(t)}{dt^2} + 6\pi\eta b\frac{d\vec{r}(t)}{dt} = \vec{F}(t)\ .$$

a) For a single realization of these random forces, express the velocity $\vec{v}(t) = \frac{d\vec{r}(t)}{dt}$ as an integral involving the past forces, $\vec{F}(t')$ for $t' < t$.

b) Find the coefficient C as a function of the temperature T and the other constants mentioned above.

c) Calculate the mean square displacement $\langle|\vec{r}(t_1) - \vec{r}(t_2)|^2\rangle$ as a function of the time difference at large time difference $|t_2 - t_1|$, and explain how measuring this quantity can be used to determine Boltzmann's constant k_B, assuming that you also have measurements of all the constants above except for k_B and C.

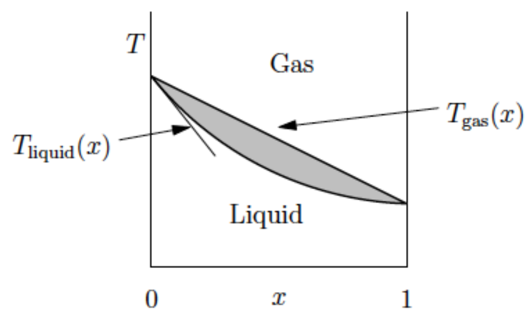
3. **Distillation**

Sketched below is the phase diagram of a mixture of two pure substances, A and B, at pressure $P = 1$ atm, where x is the mass fraction of A: $x = M_A/(M_A + M_B)$, and M_n is the mass of substance n. Assume that in the regime of interest, the boundaries of liquid-gas two-phase coexistence are given by the linear functions:

$$T_{gas}(x) = T_0 - T_1 x \ ,$$

$$T_{liquid}(x) = T_0 - 3T_1 x \ .$$

The shaded region indicates liquid-gas phase coexistence.



An open beaker initially contains a liquid mixture of total initial mass M, with A having initial mass fraction $x_i = 0.2$. The liquid is brought to its boiling temperature.

a) Does the boiling increase or decrease the mass fraction of A in the liquid?

b) The boiling is continued until the mass fraction x of A in the liquid is changed by a factor of two. At this point, what fraction of the initial total mass M remains? How does the mass fraction x of A change as a function of the total mass of the liquid remaining?