Department of Physics, Princeton University

Graduate Preliminary Examination
Part I

Thursday, January 7, 2016
9:00 am - 12:00 noon

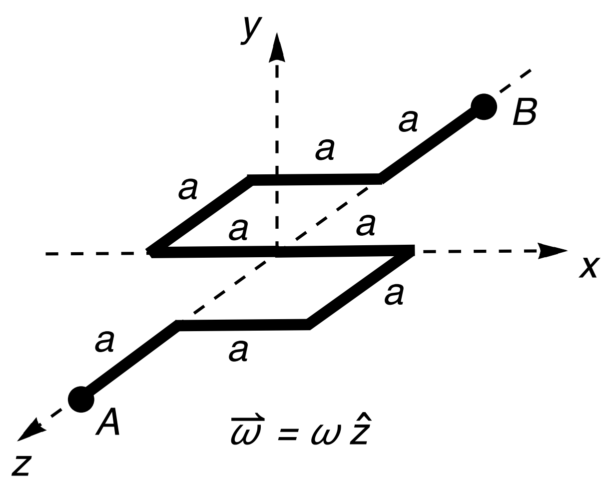
Answer TWO out of the THREE questions in Section A (Mechanics) and TWO out of the THREE questions in Section B (Electricity and Magnetism).

Work each problem in a separate examination booklet. Be sure to label each booklet with your name, the section name, and the problem number.

Section A. Mechanics

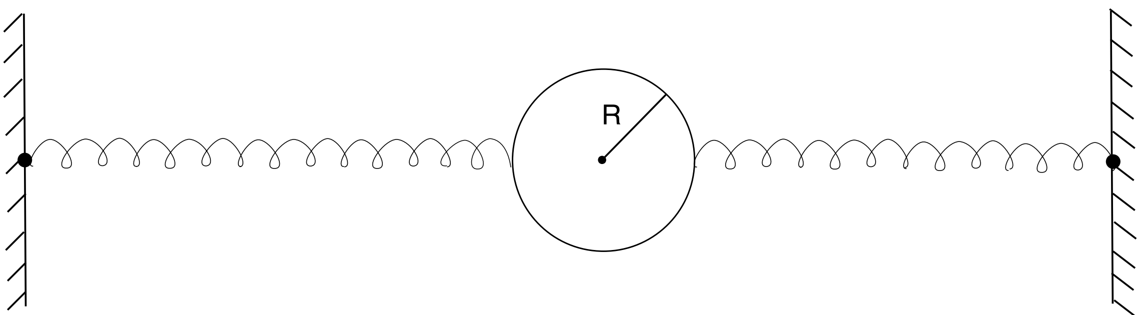
1. **Rotating Crankshaft**

   An automobile crankshaft is a planar rigid body made of 8 rods each of mass m, length a, welded together as shown. Suppose the crankshaft rotates about the z axis with constant angular velocity $\omega > 0$. Find the directions and magnitudes of the forces on the two bearings A and B at a moment when the crankshaft lies in the $x - z$ plane as shown. The bearings are located on the ends of the two rods which lie along the x axis. Ignore gravity.



2. **Disc on Springs**

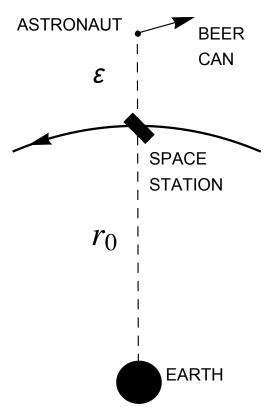
A thin uniform disk of mass M and radius R is connected by two springs of spring constant K to two fixed points on a frictionless table top. The springs are attached to the disc at opposite ends of a diameter and the disk is free to translate and rotate in the plane. Each spring has an unstretched length l_0, and when the disc sits at rest in the equilibrium position (as in the figure), both springs are stretched to the same length $l > l_0$. The motion of the disc in the plane has three degrees of freedom which we can take to be the coordinates (x, y) of the center of the disc and the angle ϕ of rotation of the disc with respect to its orientation when at rest.



What are frequencies of the normal modes of oscillation for small motions about this equilibrium position?

3. **Orbiting Beer Can**

A space station is in a circular orbit about the earth at a radius r_0. An astronaut on a space walk happens to be distance ε from the station on the line joining the station to the center of the earth. With practice the astronaut can throw a beer can so that it appears to orbit the space station, in the plane of the space station's orbit about the earth. We can neglect the gravitational attraction between the beer can and the space station, so the beer can is just orbiting the Earth in a slightly non-circular orbit, cleverly chosen so that the beer can never drifts away from the space station as the latter moves in its own circular orbit. Let's work out the details of this trick.



(a) First a helpful lemma: Use the radial equation for orbits of a given specific angular momentum $\hat{\ell} = \ell/m$ to show that the period of *small* radial oscillations about the circular orbit is the same as the period of the circular orbit itself.

(b) The astronaut launches the beer can with zero radial component of velocity, and with tangential velocity chosen so that its specific angular momentum $\hat{\ell} = \ell/m$ equals that of the space station. Find the subsequent motion of the beer can in Earth-centered radial coordinates $(r(t), \theta(t))$, working to the first approximation in small deviations from a circular orbit.

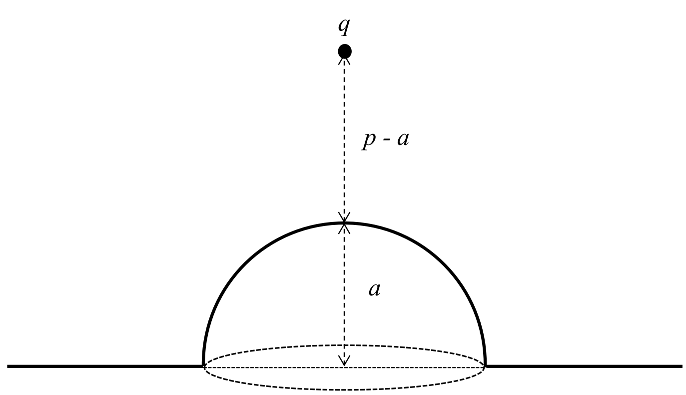
(c) Construct the orbit of the beer can relative to the space station, using x-y coordinate axes centered on the station, rotating with the station, so that the negative y-axis always points down toward the center of the Earth. Show that the orbit in these station-centric coordiantes is an ellipse and give its period, semi-major and semi-minor axes.

Section B. Electricity and Magnetism

1. **Conducting plane with bulge**

   (a) A spherical conductor of radius a is at potential $V = 0$ with respect to infinity. A charge $Q = q$ is brought to a distance $p > a$ from the center of the sphere and you are asked to find the force on the charge. Show that this can be determined with the help of a notional image charge $Q' = -\frac{a}{p}q$ located a distance $\frac{a^2}{p}$ from the center of the sphere.

   (b) Use what you have learned in a) about image charges in a sphere to analyze the following more complicated situation: A conductor at potential $V = 0$ has the shape of an infinite plane except for a hemispherical bulge of radius a. A charge q is placed above the center of the bulge, a distance p from the place (distance $p - a$ from the top of the bulge). What is the force on the charge?



2. **Long Antenna Pattern**

A thin, straight, conducting wire centered on the origin and oriented along the z-axis carries a current

$$I = \hat{z} I_0 \cos \omega t$$

everywhere along its length ℓ. This antenna will radiate electromagnetic waves with frequency ω and wavelength $\lambda = 2\pi c/\omega$. We will not assume that $\ell << \lambda$.

(a) Because of the current, a time-dependent charge $q(t)$ will accumulate at the two ends of the wire. Give expressions for the charge and current *densities* $\rho(\vec{x}, t)$ and $\vec{j}(\vec{x}, t)$ on the wire. Show that the electric dipole moment of this charge distribution satisfies $p(t) = p_0 \sin(\omega t)$ and evaluate p_0

(b) Use these source densities to construct the scalar and vector potentials everywhere outside the source region ($r \gg \ell$). Do not assume anything about the relative magnitudes of ℓ and λ. Do state the gauge you are using.

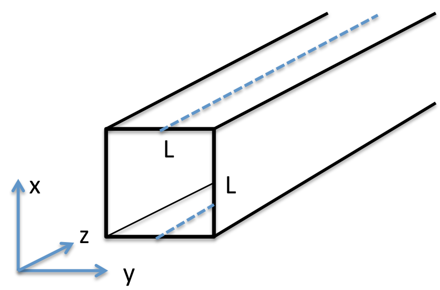
(c) Compute the angular distribution of the energy flux radiated from this antenna. Show that it reduces to the standard electric dipole radiation pattern when $\lambda \gg \ell$. For general λ, show that the energy flux radiated perpendicular to the $\hat{z}$ direction depends only on the maximum electric dipole moment p_0 (and agrees with the standard electric dipole radiation result).

3. **Slicing a Waveguide**

A square waveguide with perfectly conducting walls at $x = 0$, $x = L$, $y = 0$, and $y = L$ extends along the z-axis. A waveguide like this has many modes that propagate in the z-direction. Consider the restricted set of propagating modes for which the E-field has only an x-component:

$$\vec{E} = (E_x, 0, 0) \qquad E_x = E_0 \sin(m\pi y/L) \sin(kz - \omega t) \quad m = 1, 2, \ldots$$

and the B-field only has components in the orthogonal directions, $\vec{B} = (0, B_y, B_z)$.



(a) Show that the specified form of the E-field satisfies the vacuum Maxwell equation $\vec{\nabla} \cdot \vec{E} = 0$ inside the wave guide, and also satisfies the appropriate boundary condition at the conducting walls.

(b) Write down expressions for the $B_{y,z}(y, z, t)$ components that must accompany this E-field in order to satisfy the vacuum Maxwell equation $-\partial\vec{B}/\partial t = \vec{\nabla} \times \vec{E}$. Show that they satisfy the appropriate boundary conditions at the conducting walls.

(c) For each of these modes, find the dispersion relation $\omega(k)$ that guarantees that all the vacuum Maxwell equations are satisfied.

(d) The cavity is now sawed in half on the plane $y = L/2$, but not pulled apart (along the dotted lines in the figure). This means that no surface currents can flow across the cut line. What are the allowed mode frequencies now? What is the answer to this question if the cut is made along $x = L/2$?

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Graduate Preliminary Examination
Part II

Friday, January 8, 2016
9:00 am - 12:00 noon

Answer TWO out of the THREE questions in Section A (Quantum Mechanics) and TWO out of the THREE questions in Section B (Thermodynamics and Statistical Mechanics).

Work each problem in a separate booklet. Be sure to label each booklet with your name, the section name, and the problem number.

Section A. Quantum Mechanics

1. **Perturbed Harmonic Oscillator (MR solution)**

   A particle of mass m moves one-dimensionally in a static harmonic oscillator potential

   $$V = \frac{1}{2} m\omega^2 x^2$$

   It is also acted on by a space-time dependent perturbation potential $W(x,t)$ that is narrowly localized around a point $x_0(t)$ in space that moves with time. To simulate this, take the delta function expression

   $$W = \lambda\delta(x - x_0(t))$$

   where λ parametrizes the potential strength.

   Let $x_0(t) = vt$ for some velocity v and suppose that the particle was in the oscillator ground state $u_0(x)$ in the remote past (at time $t \to -\infty$). What is the probability that the particle will be found in the first excited oscillator state $u_1(x)$ in the remote future? Treat W as a small perturbation and work out the answer to lowest order in λ. Sketch the dependence of the transition probability on v and identify the value of v that maximizes the transition probability.

   You are reminded that

   $$u_0 = \left(\frac{1}{\pi a^2}\right)^{1/4} \exp(-x^2/2a^2), \qquad u_1 = \sqrt{2}\frac{x}{a} u_0, \qquad a = \sqrt{\frac{\hbar}{m\omega}}.$$

2. **Decay Angular Correlations**

An unpolarized nucleus of spin S (to be determined) decays into a nucleus of spin 0, plus two alpha particles. The alpha particles have spin 0 and there are of course many possibilities for their orbital angular momentum. Let us consider the case that both have orbital angular momentum $L = 1$. By angular momentum addition, the original nucleus could have had $S = 2, 1, \text{or}\, 0$. We can distinguish the three cases by measuring the probability distribution of the angle between the directions of motion of the outgoing alphas. Since the original nucleus is unpolarized, there is no other meaningful angle in the problem.

(a) As a first step, use the techniques of angular momentum addition to construct states of total angular momentum 2 out of two particles of orbital angular momentum 1; that is, find the linear combinations of $Y_{1m_1}(\theta_1, \phi_1)Y_{1m_2}(\theta_2, \phi_2)$ that transform in the angular momentum 2 representation.

(b) Next, compute the probability distribution of the angle between the two alphas in the case that the original $S = 2$ nucleus is unpolarized (*i.e.* has equal *probability* of being in the 5 different S_z substates). Work is simplified, at no cost in generality, by assuming that both alphas lie in the plane perpendicular to the quantization axis (so that in your spherical harmonics $\theta = \pi/2$ and only ϕ varies).

(c) Next, do the same computation for the case that the intial nucleus has $S = 0$ and compare with b) to show that the two cases can be distinguished.

(d) Why do we not ask you to consider the case $S = 1$?

You will need the $L = 1$ spherical harmonics:

$$Y_{10}(\theta, \phi) = \sqrt{\frac{3}{4\pi}} \cos\theta \qquad Y_{1\pm1}(\theta, \phi) = \sqrt{\frac{3}{8\pi}} \sin\theta e^{\pm i\phi}$$

3. **Magnetic Barrier**

Consider the quantum mechanics of a charged particle moving in the $x-y$ plane subject to a magnetic field $B_z = B_0\theta(x)\theta(d-x)$. In other words, the magnetic field is constant in a strip of width d and zero everywhere else. Including a vector potential in the Schrödinger equation is very simple: just replace ∇ in the kinetic energy by $\nabla - \frac{ie}{\hbar c}\vec{A}(\vec{x})$. We will study the problem of scattering from this 'barrier' of an electron incident from $x < 0$ with momentum parallel to the x-axis. Note: you must choose a gauge for the vector potential describing the B field — the gauge in which $A_x = A_z = 0$ everywhere, with only $A_y(x)$ non-vanishing is particularly convenient here.

(a) For an incident wave $\exp(ikx)$ there will, in general, be a transmitted wave $T\exp(i\widehat{k}x)$ and a reflected wave $R\exp(-ikx)$. Show how the transmitted wave vector $\widehat{k}$ is determined by k and B_0d.

(b) For a given choice of B_0d, you will find that, below a certain critical energy E_0, $\widehat{k}$ is imaginary. Show that a classical description of how an electron moves in the presence of a magnetic strip leads to the same critical energy.

(c) When $\widehat{k}$ is real, there is a transmitted wave that propagates to $x = +\infty$. Calculate the transmitted probability flux and show that, despite the fact that the wave function depends only on x, the flux is *not* along the x-axis! Give a classical interpretation of this quantum fact.

(d) Show that in the limit $d \to 0$, $B_0 \to \infty$, with B_0d fixed, this effectively one-dimensional scattering problem can be solved exactly, and find the reflection and transmission coefficients R and T.

Section B. Statistical Mechanics and Thermodynamics

1. **String Thermodynamics**

   An elastic string is found to have the following properties:

   - To stretch it to a total length x requires a force $f = \mu x - \alpha T + \beta T x$. Assume that α, β, μ are constants.
   - Its heat capacity at constant length x is proportional to temperature: $C(x) = A(x)T$.

   We can use thermodynamic identities to derive from these facts a variety of other thermal properties. More specifically:

   (a) Calculate $\frac{\partial S}{\partial x}|_T$.

   (b) Show that A has to be independent of x.

   (c) Calculate $\frac{\partial S}{\partial T}|_x$ and give the general expression for entropy $S(x,T)$ assuming $S(0,0) = B$, where B is a constant.

   (d) Compute the heat capacity at zero tension $C_F = T\frac{\partial S}{\partial T}|_{f=0}$.

2. **Ideal Gas with Funny Dispersion Relation**

Consider a gas of non-interacting particles with no internal degrees of freedom confined to a three-dimensional box of volume V, and obeying the dispersion relation $\epsilon(\vec{k}) = A\,|k|$. We want to study the dilute gas equation of state for this gas, taking account of quantum statistics.

(a) Let $N(\epsilon) = VH(\epsilon)$ be the total number of single particle states in the box with energy less than ϵ. Define the density of states by $dN(\epsilon)/d\epsilon = VG(\epsilon)$. Calculate $H(\epsilon)$ and $G(\epsilon)$ for the given dispersion relation.

(b) Use the grand canonical ensemble to derive the standard expressions for pressure and density :

$$n = \int_0^\infty d\epsilon \frac{G(\epsilon)}{(e^{\beta(\epsilon-\mu)} \mp 1)} \qquad p = \int_0^\infty d\epsilon \frac{H(\epsilon)}{(e^{\beta(\epsilon-\mu)} \mp 1)}$$

where μ is the chemical potential and the minus (plus) signs correspond to Bose (Fermi) statistics.

(c) Show that in the limit of large negative μ (low density) the pressure satisfies the ideal gas equation of state (independent of statistics).

(d) Finally, compute the first non-trivial correction (in powers of density) to the ideal gas equation of state and determine how it depends on density and temperature for both bosons and fermions.

3. **Hydrogen Recombination**

In this problem we investigate the formation of hydrogen atoms in the early universe. Although the binding energy of hydrogen is 13.6 eV, the majority of protons and electrons did not become bound into atoms until the temperature of the neutral primordial plasma cooled to a much lower temperature, about 0.3 eV. To study this problem, we make four assumptions:

- The hydrogen atom has no bound states apart from its ground state.
- We ignore other bound complexes that might be formed, e..g, hydrogen ions and molecules.
- All interactions among hydrogen atoms protons and free electrons are ignored (apart from the fundamental process of atom formation).
- Everything is in thermal equilibrium.

There are two questions:

(a) Assume that at $T = 0.3$ eV half of the protons had a bound electron. From this information calculate the densities (in units of particles per cubic centimeter) of free electrons, free protons, and hydrogen atoms.

(b) At $T = 0.3$ eV what is the density of photons? How does the photon density compare to the total density of baryons (protons plus hydrogen atoms) obtained in part (a)? You have just calculated the photon to baryon ration in our universe, starting from a remarkably simple piece of information.

The following constants will be useful to know:

$$hc = 2.0 \times 10^{-25}\,\mathrm{Jm} = 1.3 \times 10^{-6}\mathrm{eVm}, \quad m_{electron} = .5 \times 10^{6}\mathrm{eV}$$

It will also be useful to know the following integral:

$$\int_0^\infty du u^2 \frac{1}{e^u - 1} = 1.4$$