

M05M.1 - Vibration Damping With a Piston

Problem

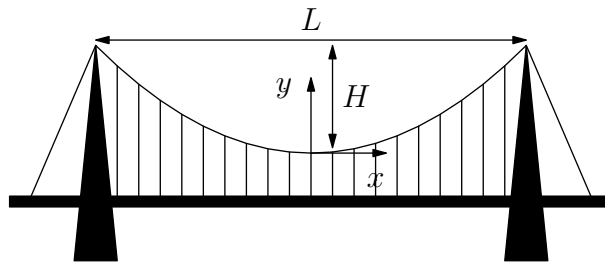
Sometimes it is required to reduce the influence of vertical floor vibration on experimental setups. To accomplish this, the experimental apparatus (of mass M) is mounted on gas-filled pistons (which have face area A and equilibrium volume V_0). The gas pressure in the cylinder is much larger than atmospheric pressure. The pistons are typically equipped with a mechanism to dampen their oscillations (by immersing part of the piston in oil), so that there is a retarding force, $-vM/\tau$, where v is the velocity of the piston and τ is the characteristic damping time.

- a) Ignoring damping ($\tau \rightarrow \infty$), determine the resonance frequency ω_0 of the system for small vibrations. Explain whether any changes in state of the gas are more nearly isothermal or adiabatic.
- b) Characterize the performance of this vibration-isolation system as a function of frequency by calculating its transmission coefficient, i.e. the ratio of the vibration amplitude of the apparatus to a small vibration in the floor. Include any phase shifts and express your answer in terms of ω_0 and the damping time τ . For which frequency range do you expect this system to be useful for isolating the vibrations of the floor?
- c) How should the system parameters be chosen to provide as much vibration isolation as possible, subject to the constraint that a sudden displacement of the floor causes no oscillations in the system?

M05M.2 - Suspension Bridge Shape

Problem

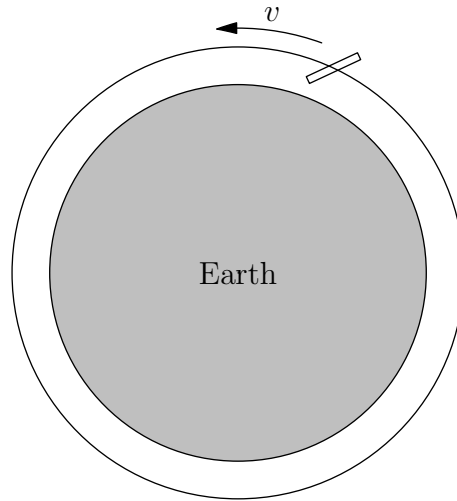
Determine the equilibrium shape $y(x)$ of the suspension cable in a bridge shown in the figure. The length of the roadbed is L and the mass M is so large that you can neglect the weight of the suspension cable and the and the vertical supporting cables. The lengths of the vertical cables are adjusted so there is no shear stress in the roadway. You can also assume that the vertical cables are close enough together that y can be approximated as an analytic function of x .



M05M.3 - Satellite Orientation

Problem

The orientation of a satellite in low-Earth orbit can be stabilized with gravity gradients. Let the satellite be a long cylinder of length L and radius a ($a \ll L$). The distribution of the mass m is uniform within the cylinder. The satellite is in a circular orbit around Earth with a period T_{or} . Express your answers to parts b), c) and d) in terms of T_{or} , L and a .



- What is the stable equilibrium orientation, for which the satellite appears to be at rest in a coordinate system rotating with the same angular velocity as the orbital motion?
- What is the period T_π of small oscillations about equilibrium in the orbital plane?
- What is the period T_σ of small oscillations about equilibrium perpendicular to the orbital plane?
- The satellite is given a small angular speed ω around its long axis. To first order in ω , find the new stable-equilibrium orientation of the satellite, where the direction of the long axis appears to be fixed in the rotating coordinate system.

M05E.1 - Periodic Surface Charge and Magnetization

Problem

- a) A thin sheet lies in the x - y plane and carries a periodic surface charge density given by $\sigma = \sigma_0 \sin(kx)$. Calculate the electric field produced by this charge distribution everywhere in space.
- b) A periodic magnetization pattern is written onto a magnetic tape of thickness w . The magnetization is perpendicular to the plane of the tape (the x - y plane) and is given by $M_z = M_0 \sin(kx)$ for $|z| \leq w/2$ and $M_z = 0$ otherwise. Calculate the magnetic field inside and outside of the tape for distances close to the surface of the tape. Here the edges of the tape can be considered at infinity.

M05E.2 - Coaxial Cable Motion

Problem

A long, straight coaxial cable of length L has an inner conductor of radius a and an outer conductor of inner radius b . Assume the insulating region between the conductors has free-space values of the electric and magnetic permittivities. One end of the cable is attached to a load resistor R . At the other end of the cable a battery of voltage V is suddenly attached. Just before the current starts to flow, the battery, cable and resistor are all at rest with no external forces. Their total mass is M . After some time, a steady-state current flows along the inner conductor and returns along the outer one. The current may be taken as uniformly distributed throughout the inner conductor, which has conductivity σ . The resistance of the outer conductor can be neglected, so the return current flows as a thin surface sheet of radius b .

- What is the electromagnetic momentum per unit length in the coax?
- What is the mechanical momentum of the cable with its attached battery and resistor after the steady-state current has been established.
- What is the charge per unit length on the outer conductor?

Possibly useful identity for cylindrical coordinates ρ, z and θ :

$$\nabla^2 = \frac{1}{\rho} \frac{\partial}{\partial \rho} \rho \frac{\partial}{\partial \rho} + \frac{1}{\rho^2} \frac{\partial^2}{\partial \phi^2} + \frac{\partial^2}{\partial z^2}$$

M05E.3 - Electron Orbit Decay

Problem

In the Bohr model of the hydrogen atom's ground state, the electron moves in a circular orbit of radius $a_0 = 0.53 \times 10^{-10}$ m around the proton, which is assumed to be rigidly fixed in space. Since the electron is accelerating, a classical analysis suggests that it will continuously radiate energy, and therefore the radius of the orbit would shrink with time.

- a) Assuming that the electron is always in a nearly circular orbit and that the rate of radiation of energy is sufficiently well approximated by classical, nonrelativistic electrodynamics, how long is the *fall time* of the electron, i.e. the time for the electron to spiral into the origin?
- b) The charge distribution of a proton has a radius of about 10^{-15} m, so the classical calculation would be modified once the radius of the electron's orbit is smaller than this. But even before this, modifications may be required due to relativistic effects. Based on the analysis of part a), at what radius of the electron's orbit would its velocity be, say $0.1c$, where c is the speed of light, such that relativistic corrections become significant? What fraction of the electron's fall time remains according to part a) when the velocity of the electron reaches $0.1c$?
- c) Do the relativistic corrections increase or decrease the fall time of the electron?

M05Q.1 - Hydrogen Atom in an Electric Field

Problem

- a) A spatially uniform, time-independent electric field \mathbf{E} is applied to a hydrogen atom for which the electron was initially in its ground state $\phi_g = e^{-r}/\sqrt{\pi}$. The field is much too small to ionize the atom in any reasonable time, but it slightly distorts the electron charge distribution. The distorted electron wave function is $\psi = \psi(\mathbf{r})$, where \mathbf{r} is the displacement (in units of the Bohr radius a_B) of the electron from the nucleus. You can ignore the spins of the electron and the nucleus. Since the polarized atom experiences no net force in the uniform field \mathbf{E} , the quantum-mechanical expectation value $\mathbf{E}' = (e/a_B^2)\langle\psi|e\mathbf{r}/r^3|\psi\rangle$, of the electric field produced at the nucleus by the distorted electron charge distribution must cancel the applied field. Prove that this is so.
- b) The applied electric field oscillates along the z axis with amplitude $E \cos \omega_{eg}t$, where $\omega_{eg} = (E_e - E_g)/\hbar$ is the Bohr frequency of the transition from the $1s$ ground state, of energy E_g , to the $2p$ excited state, of energy E_e . Find \mathbf{E}' in this case. Assume that $eEa_B \ll \hbar/\tau$, where the natural radiative lifetime of the $2p$ state is given by

$$\frac{1}{\tau} = \frac{8\pi e^2 \omega_{eg}^3 a_B^2 z_{eg}^2}{3\hbar c^3}$$

The matrix element of electron displacement along the z axis is,

$$z_{eg} = \int \phi_e^* z \phi_g d^3r,$$

and e is the elementary charge.

HINT: One of these problems has a solution that does not rely on perturbation theory.

Possibly useful hydrogen wave function (in atomic units) for the first excited p state of axial symmetry:

$$\psi_e = \frac{ze^{-r/2}}{\sqrt{32\pi}}$$

Possibly useful definite integral for $a > 0$:

$$\int_0^\infty x^n e^{-ax} dx = \frac{n!}{a^{n+1}}.$$

M05Q.2 - Delta Function Potential

Problem

A particle of mass m and kinetic energy E scatters from a thin spherical shell of radius R . The scattering potential can be approximated by

$$V = vR\delta(r - R),$$

where r is the distance of the particle from the scattering center, v is a characteristic energy, and δ denotes a Dirac delta function.

- a) Derive the S-wave scattering cross-section σ_0 in terms of E, m, v, R and \hbar .
- b) For what energies E does σ_0 vanish? Explain.
- c) Derive a formula for σ_0 when $E = 0$.
- d) For what values of v does $\sigma_0 \rightarrow \infty$ as $E \rightarrow 0$ in c)? Explain.

M05Q.3 - Nitrogen and Explosives

Problem

The nitrogen nucleus, ^{14}N is in a chemical environment (an explosive) where the nuclear spin Hamiltonian H can be described by

$$H/\hbar = Q_x I_x^2 + Q_y I_y^2 + Q_z I_z^2 + \boldsymbol{\omega} \cdot \mathbf{I}.$$

The nuclear spin operators are I_x, I_y and I_z and the nuclear spin quantum number is $I = 1$.

The interaction of the nuclear electric quadrupole moment with the electric-field gradient at the site of the nucleus is characterized by the parameters Q_x, Q_y and Q_z . The coordinate system is chosen so that the gradient tensor is diagonal. The energy scale is chosen so that $Q_x + Q_y + Q_z = 0$.

The Larmor frequency due to the earth's magnetic field is ω , with components ω_x, ω_y and ω_z along the principal axes of the field-gradient tensor. You can assume that $\omega_x^2 + \omega_y^2 + \omega_z^2 \ll Q_x^2 + Q_y^2 + Q_z^2$.

- a) For $\omega = 0$, find the energy eigenvalues and eigenfunctions for the nucleus.
- b) For sufficiently symmetric environments, for example, for the anion NO^{3-} , the field gradient is axially symmetric and can be characterized by a single frequency parameter Q , with $Q_x = Q_y = -Q$ and $Q_z = 2Q$. To lowest nonvanishing order in $\omega \neq 0$, find the energy perturbations for this symmetric situation.

M05T.1 - Thermodynamics of Solid Argon

Problem

At atmospheric pressure and low temperature the specific heat of solid argon (melting point 84K) is

$$c_p = 49.9(T/\theta)^3 \text{ kJ/kg K}$$

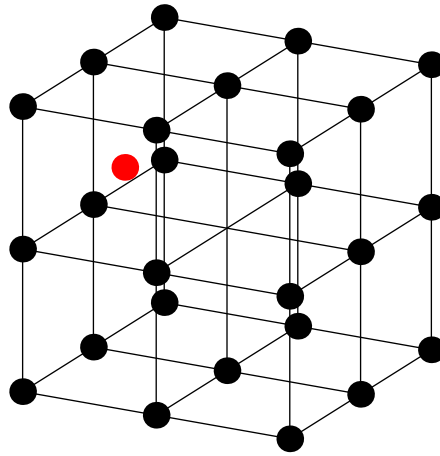
where $\theta = 92 \text{ K}$ is the Debye temperature.

- a) What is the specific entropy s as a function of T and as a function of the specific energy u ? Use Nernst's third law if needed.
- b) How much energy is needed to warm 100 g of argon from 4 K to 20 K?
- c) What is the minimum energy needed to cool it back to 4 K given that the temperature of the environment is room temperature, 20°C, and that no colder body is available?

M05T.2 - Defects in a Cubic Crystal

Problem

A cubic crystal contains N atoms. The atoms can exist at lattice sites or an atom may find itself displaced from its normal site into the center of one of the 8 adjacent unit cells as suggested by the figure. (Note that the figure shows only a small part of the crystal, N is very large.) When an atom is displaced, its energy is increased by $\epsilon > 0$ over its energy at the normal lattice site. Suppose the crystal is in equilibrium at temperature T . assume $k_B T \ll \epsilon$ so the chance that two displaced atoms try to occupy the same cell is negligible. Calculate the partition function, the entropy, the energy (relative to the energy the crystal would have if all atoms were at their normal sites), and the number of atoms in displaced sites.



M05T.3 - Surface Waves and Heat Capacity (M04T.2)

Problem

Consider waves on a liquid surface where the restoring force is produced by the surface tension. Assume there is a single polarization and the dispersion relation is

$$\omega^2 = \frac{\gamma}{\rho} k^3,$$

where γ is the surface tension of the liquid, ρ is its density, ω is the frequency of the waves and k is their wavenumber. Our goal is to find the contribution of these waves to the low temperature heat capacity of the liquid. Surface wave excitations are quantized (quasiparticles, like phonons!).

- a) If the surface is in equilibrium at temperature T , what is the average energy of a wave with frequency ω ? (Ignore the $\hbar\omega/2$ zero point energy.)
- b) At low temperatures what are the energy per unit area and heat capacity per unit area of these surface waves?