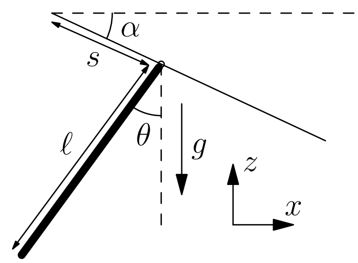
J10M.1 - Rod on a Rail (M93M.2)

Problem



A uniform rod of length ℓ and mass m moves in the x-z plane. One end of the rod is suspended from a straight rail that slopes downwards with an angle α relative to the horizontal; the connection point is free to move along the rail without friction, and the rod is able to swing freely in the x-z plane. Uniform gravity acts downwards.

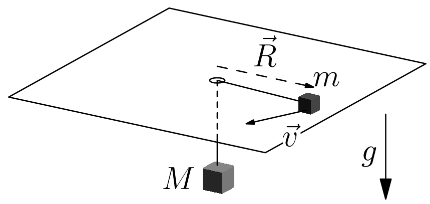
a) Construct the Lagrangian of this system in terms of generalized coordinates s (the distance the connection point has moved along the rail) and θ (the angle the rod makes with the vertical direction).

b) Using your Lagrangian, find a solution to the equation of motion where the rod moves with fixed θ as s increases.

c) Explain how your solution is consistent with (and can be derived from) the equivalence principle.

J10M.2 - Orbiting Mass on a String (J00M.3)

Problem



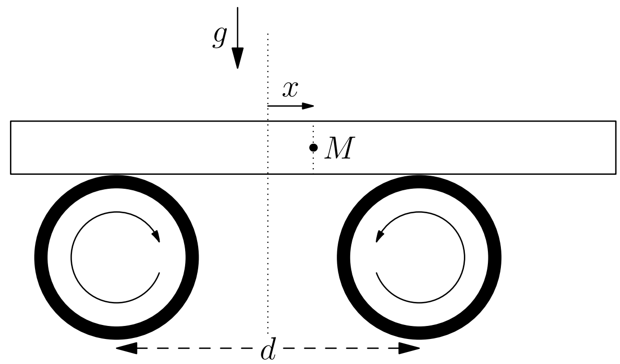
A hockey puck with mass m can move without friction or air resistance on the smooth horizontal surface of a table. A massless string attached to the puck passes through a hole in the table (through which it can slide without friction) and a mass M is suspended from its other end. Gravity acts uniformly in the downward direction. Treat the puck as a point mass.

a) Given the masses m and M, plus the initial displacement $\vec{R}_0$ of the puck relative to the hole, and its initial velocity $\vec{v}_0$ in the plane of the table surface, find the equation that determines the maximum and minimum radial distances of the puck from the hole during its orbit. (Don't bother to solve this equation!)

b) Find the frequency of oscillations of the radial distance when the orbit is close to being circular.

J10M.3 - Slab on Rotating Rollers

Problem

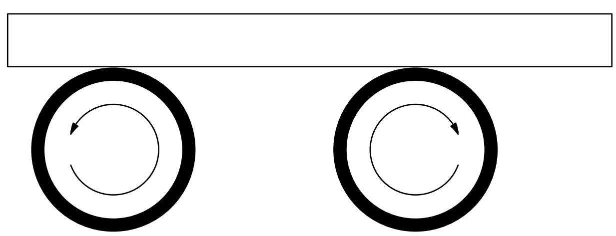


A uniform rigid slab of mass M is supported by two rapidly counter-rotating parallel horizontal rollers, with axes a distance d apart, with surfaces that brush past the slab in the directions shown in the figure. The coefficient of kinetic friction between each roller and the slab is μ_k.

At time $t = 0$, the center of mass of the slab is initially displaced horizontally by $x(0) = x_0$ (where $|x_0| < d/2$) relative to the midpoint between the rollers, and the slab is initially at rest, $\dot{x}(0) = 0$.

a) Write down the equation of motion for $x(t)$, and solve it for $t > 0$ with the given initial conditions.

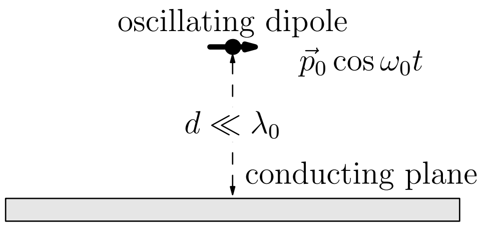
Now consider the case where the directions of the rollers are reversed, as shown below:



b) Calculate $x(t)$ for $t > 0$ for the same initial conditions, in this second case.

J10E.1 - Oscillating Dipole Near a Conducting Plane

Problem



An electric dipole is forced to oscillate with frequency ω_0 and amplitude $\vec{p}_0$, so $\vec{p}(t) = \vec{p}_0 \cos \omega t$. It is placed in vacuum at a distnace $d \ll c/\omega_0 = \lambda_0$ away from an infinite perfectly-conducting plane, with $\vec{p}_0$ parallel to the plane. The physical dimensions of the dipole are infinitesimal compared to compared to d, and it can be treated as a point dipole.

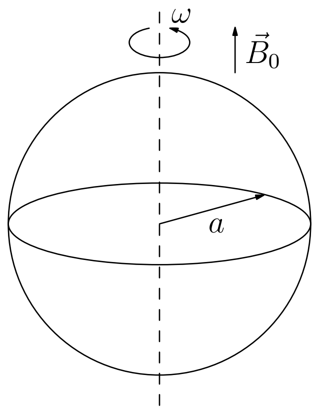
At distances from the dipole that are large compared to λ_0:

a) Find the steady-state electromagnetic fields $\vec{E}(\vec{r}, t)$ and $\vec{B}(\vec{r}, t)$.

b) Find the angular distribution of the radiated power of the emitted radiation.

J10E.2 - Rotating Sphere in a Magnetic Field

Problem



A solid metallic sphere of radius a has finite conductivity, carries no net electric charge, and is free to rotate without friction about a vertical axis through its center. The region outside the sphere is vacuum. There is a uniform magnetic field with flux density $\vec{B}_0$ parallel to the axis.

The sphere is given an impulse that starts it spinning around the axis and there is some initial Ohmic dissipation. After the dissipation has ceased, the sphere is in a steady state of rigid rotation with constant angular velocity ω_∞.

In steady state, *to lowest order in both* B_0 *and* ω_∞, find:

a) The electric field $\vec{E}(\vec{r})$ and electric potential $\Phi(\vec{r})$ in the *interior of the sphere*, $r < a$. (Give these in the non-rotating "laboratory frame".)

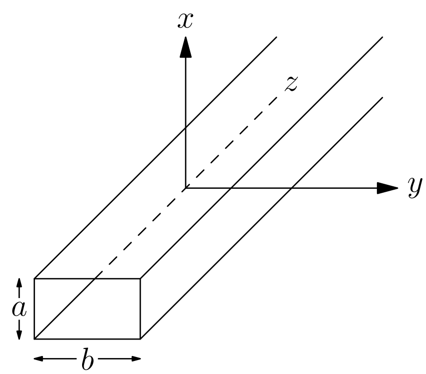
b) The electric potential *outside* the sphere. (Express your answer in spherical coordinates (r, θ, ϕ).) State the nature of the electric field it describes (*i.e.*, monoole, dipole, quadrupole, *etc.*).

c) The induced bulk and surface charge density distributions in the conductor that give rise to this potential.

Note: By working to lowest order in B_0 and ω_∞, you can ignore both the mechanical deformation of the metal sphere due to rotation and the magnetic fields generated by currents in the metal (therse are negligibly small relative to B_0).

J10E.3 - Rectangular Waveguide

Problem



A transverse electric (T.E.) wave is propagating in an infinitely long rectangular waveguide with perfectly conducting walls. The waveguide is filled with a dielectric (dielectric constant ϵ and relative magnetic permeability $mu = 1$). The electric field inside it is

$$E_x = E_0 \sin\left(\frac{\pi y}{b}\right) e^{i(kz-\omega t)}\,, \quad E_y = 0\,.$$

a) Find the corresponding $\vec{B}$ field.

Suppose now that the dielectric is removed from the region $z > 0$ inside the waveguide, so it is vacuum. The region $z < 0$ remains filled with dielectric, as before, and the electric field of the incident wave in the region $z < 0$ is that given above.

b) Find the transmitted $\vec{E}$ field in the vacuum region $z > 0$.

c) For what range of ω will there be no transmitted propagating wave in the vacuum region $z > 0$?

J10Q.1 - Harmonic Oscillator

Problem

Consider an isotropic three-dimensional harmonic oscillator described by the rotationally-invariant Hamiltonian

$$H = \frac{|\vec{p}|^2}{2m} + \frac{m\omega^2}{2}|\vec{v}|^2 \, .$$

a)

i. What are the energies and degeneracies of the lowest three energy levels?

ii. Account for the degeneracies by classifying states in these levels into total angular momentum multiplets.

b) By how much does the ground state energy change under the influence of a perturbation of the form

$$H' = \lambda(\vec{b} \cdot \vec{x})^3$$

where $\vec{b}$ is some fixed vector, and λ is small? Calculate the correction up to second order in λ.

Now suppose that the oscillating particle has charge q. At time $t = 0$, a weak uniform electric field $\vec{E}$ is switched on, which then slowly decays as $\vec{E}(t) = \vec{E}_0 e^{-t/\tau}$, with $\tau > 0$.

c) What is the probability (to leading order in $|\vec{E}_0|$) that a system originally in the ground state will be in an excited state at a much larger time $t \gg \tau$?

J10Q.2 - Angular Momentum

Problem

A two-particle system is in a state $|\Psi_0\rangle$, where each particle has orbital angular momentum quantum numbers $\ell = 1$ and $m_\ell = 0$.
Let $\vec{L}_{\text{tot}} = \vec{L}_1 + \vec{L}_2$ be the total angular momentum of the two particles, where L^2_{tot} has eigenvalues $\hbar^2 L(L+1)$.

a) If the two-particle state is expanded in eigenstates of L^2_{tot}, which values of L have non-zero amplitude in the expansion? For each of these values, what is the probability that it will be found in a measurement of $|\vec{L}_{\text{tot}}|^2$?

At time $t = 0$, a coupling between the particles is “switched on”, so that for $t > 0$ the time evolution of the state is governed by the Hamiltonian

$$H = \gamma \vec{L}_1 \cdot \vec{L}_2 \,.$$

The amplitude $f(t) = |\langle \Psi(t)|\Psi_0\rangle|^2$ oscillates as a function of time, returning to the value 1 at times $t = t_n = nT$. What is the period T?

b) What is the value of $f(t)$ when $t = (t_n + t_{n+1})/2$?

J10Q.3 - Spin-Dependent Scattering

Problem

Consider the usual Hamiltonian for non-relativistic electrons moving in 2D:

$$H = H_0 + V(x,y)\,, \quad \text{where} \quad H_0 = \frac{p_x^2}{2m} + \frac{p_y^2}{2m}\,.$$

Electrons experience a "step" potential $V(x,y) = 0$ for $x < 0, V = V_0 > 0$ fro $x > 0$.

a) Electrons arriving from the region $x < 0$ are incident normally in the step (*i.e.*, have conserved momentum $p_y = 0$). Find the probability of reflection.

Now consider a similar problem, but this time one where the Hamiltonian couples the spatial and spin degrees of freedom of the electron in an essential way:

$$\boldsymbol{H} = \boldsymbol{H}_0 + V(x,y)\boldsymbol{\sigma}^0\,, \quad \text{where} \quad \boldsymbol{H}_0 = v_F(\boldsymbol{\sigma}^x p_y - \boldsymbol{\sigma}^y p_x)\,,$$

where $\boldsymbol{\sigma}^i$ are 2×2 Pauli matrices and $\boldsymbol{\sigma}^0$ is the identity matrix; v_F is a characteristic speed, and $V(x,y)$ is the same "step" potential as in a), above.
In this problem, eigenstates of $\boldsymbol{H}_0$ with momentum $\boldsymbol{p}$ have energie $\varepsilon_\pm(\boldsymbol{p}) = \pm v_F|p|$. They are non-degenerate for $\boldsymbol{p} \neq 0$.

b) Find the two-component wavefunction $\Psi_\sigma(x,y)$ of *positive energy* eigenstates of $\boldsymbol{H}_0$ with momentum $\boldsymbol{p} \equiv (p_x, p_y) = |p|(\cos\theta, \sin\theta)$. (The index σ labels the two possible values of the z-component of spin.)

c) Electrons arriving from the region $x < 0$ with $p_y = 0$ (as in part a)) now have 100% probability of transmission through the step. Explain why.

d) Consider electrons arriving with energy $E = 2V_0$ and $p_y = |p|\sin\theta$:

   i. For what range of θ is transmission through the step possible? (Hint: an analog of "Snell's law" relates angles of incidence and refraction, θ and θ'.)

   ii. In this range, find the reflection probability $R(\theta)$.

Note 1: The second Hamiltonian only requires that both wavefunction components $\Psi_\sigma(x,y), \sigma =\uparrow,\downarrow$, are continuous at the step, with no condition on their derivatives. (It describes electrons on the surface of a "topological insulator".)

Note 2: The Pauli matrices are:

$$\boldsymbol{\sigma}^x = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}\,, \quad \boldsymbol{\sigma}^y = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}\,, \quad \boldsymbol{\sigma}^z = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}\,, \quad \boldsymbol{\sigma}^0 = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}\,.$$

J10T.1 - Graphene

Problem

Graphene is a two-dimensional sheet of carbon atoms. Both electronic and phonon degrees of freedom contribute to the low-temperature specific heat per unit area. The electron states resembe the states of the massless Dirac equation, with energies

$$\varepsilon_{\pm}(\vec{p}) = \varepsilon_0 \pm v_{\mathrm{F}} p \,, \quad p \equiv |\vec{p}| \,,$$

where $\vec{P} = (p_x, p_y)$ is the analog of the momentum carried by a Dirac electron. (There are two energy bands, $\varepsilon_+(\vec{p}) \geq \epsilon_0$ and $\varepsilon_-(\vec{p}) \leq \epsilon_0$ which become degenerate at $\vec{p} = 0$). These states have a fourfold degeneracy (the usual two-fold spin degeneracy is doubled by an additional "valley" index).

a) If the Fermi energy E_{F} is $\varepsilon_0 + v_{\mathrm{F}} p_{\mathrm{F}}$, with $p_{\mathrm{F}} > 0$, what is the leading behavior of the electronic specific heat as $T \to 0$?

b) What is the low-temperature electronic specific heat when $p_{\mathrm{F}} = 0$?

(The next calculation is independent of parts a), b) above).

Recently, freely suspended graphene sheets have been studied. These have an unusual phonon spectrum: in addition to longitudinal and transverse sound waves with frequencies $\omega = v_{\mathrm{L}} q, v_{\mathrm{T}} q$ (where q is the magnitude of the wavenumber $\vec{q}$), there is an extra low-frequency mode $\omega = K q^2$ where atomic displacements are normal to the sheet.

c) Obtain the leading behavior of the phonon contribution to be specific heat as $T \to 0$.

You may express your answers in terms of the numerical constants

$$C_n^{\pm} = \int_0^{\infty} dx \frac{x^n}{e^x \pm 1} \,, \quad n > 0 \,.$$

J10T.2 - Maxwell-Boltzmann Gas

Problem

In a simple approximation often used to calculate transport properties, the statistical distribution of the velocities of molecules arriving at a point is taken to be that of the *local* equilibrium state where their most-recent collision occurred ($T, p, \langle\vec{v}\rangle$, *etc.*, are assumed to be slowly varying functions of position).

a) Using this approximation, derive the well-known estimate (due to Maxwell) of the viscosity η of a dilute classical gas of molecules with mass m, particle density $\bar{n}$, and mean free path ℓ between collisions. Assume a Maxwell-Boltzmann distribution of molecular velocities with $\langle|\vec{v} - \langle\vec{v}\rangle|^2\rangle = v_{\text{rms}}^2$.

b) If ℓ is modeled by treating monoatomic molecules as hard spheres with a finite diameter, how does the predicted viscosity vary with pressure p for low pressures at fixed temperature T? (Assume that ℓ remains smaller than the dimensions of the container.)

The Maxwell-Boltzmann gas can be viewed as the high-temperature limit of a quantum gas of non-relativistic particles. The Maxwell-Boltzmann treatment assumes that

$$\lambda(T) \ll \bar{n}^{-1/3} \ll \ell\,,$$

where $\lambda(T)$ is the *thermal de Broglie wavelength* of the particles.

c) In terms just of the three lengths $\lambda(T), \bar{n}^{-1/3}$, and ℓ, plus fundamental constants, give expressions for:

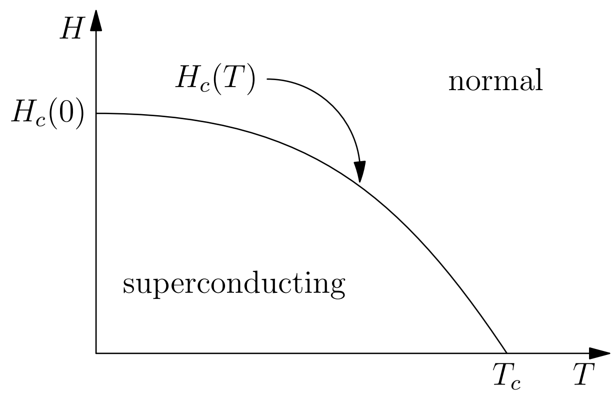
   i. The viscosity η of a Maxwell-Boltzmann gas.

   ii. The entropy density $\bar{s}$ of a monoatomic Maxwell-Boltzmann gas.

d) Estimate the lowest value that the ratio $\eta/\bar{s}$ can take before the quantum effects neglected in Maxwell-Boltzmann theory must be considered.

J10T.3 - Thermodynamics of Superconductors (O85T.1)

Problem



In the absence of a magnetic field ($H = 0$), an isotropic metal has a continuous transition to a superconducting state below a critical temperature T_c. The metal has specific heat (per unit volume) $c_V^n = \gamma T$, while in the superconductor $c_V^s = \alpha T^3$. Assume that the volume of the material does not vary with the temperature and magnetic field.

a) Find T_c as a function of γ and α.

b) For $H = 0$, give expressions in terms of T, T_c, and γ for (and sketch versus T):

i. the free energy density,

ii. the entropy density

iii. the specific heat.

In finite magnetic field strength $H > 0$, the transition becomes first order. The superconductor exhibits the Meissner effect which excludes magnetic flux density B from its interior, so $B = 0$ even though $H > 0$. Above a critical field $H_c(T)$, superconductivity breaks down, and the system becomes normal with $B = \mu H$ (to a good approximation, μ is equal to the vacuum permeability). The phase diagram is depicted above.

c) On general grounds, why must $dH_c(T)/dT$ vanish as $T \to 0$?

d) Find an expression for $H_c(T)$. (Assume that c_V^n and c_V^s do not depend on H.)

Note: When the internal energy U is defined to include the integrated electromagnetic energy density inside the material, H is a thermodynamic analog of the pressure:

$$H \equiv V^{-1}\, \partial U / \partial B|_{SVN}\,.$$