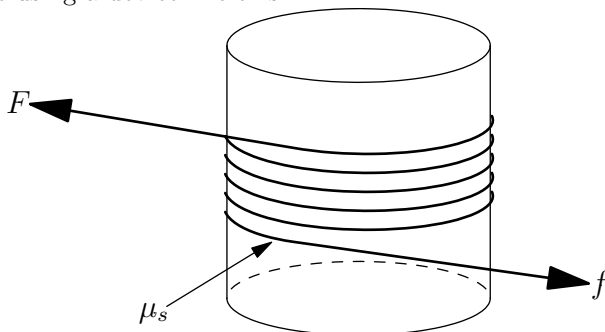


## J05M.1 - Rope Around a Cylinder

### Problem

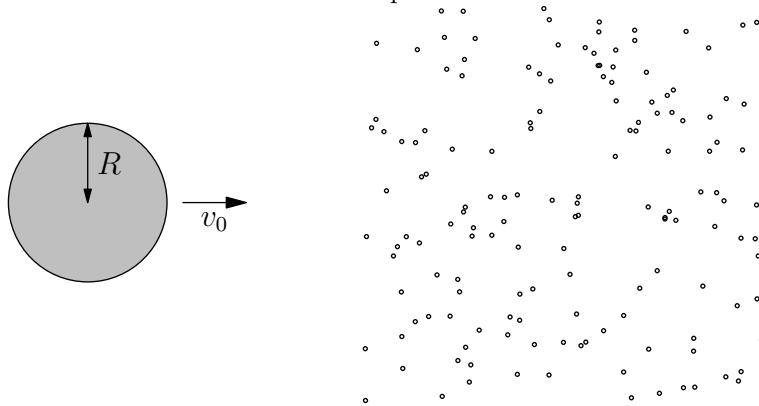
A long rope is wound around a cylinder of radius  $r$  so that a length,  $l$ , of the rope is in contact with the cylinder. The coefficient of static friction between the rope and the cylinder is  $\mu_s$ . A force  $F$  is exerted on one end of the rope. For a given  $F, r, l$  and  $\mu_s$ , what force  $f$  must be applied to avoid the rope slipping? Explain why a small child can hold a large ocean liner in place using a device like this.



## J05M.2 - Planet Moving Through Dust Cloud

### Problem

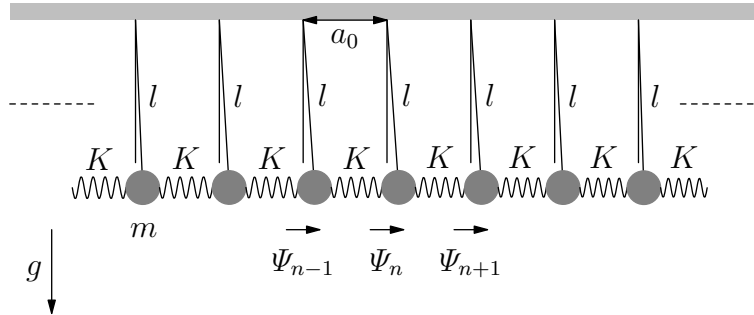
A planet of mass  $M$  and radius  $R$  moves through a cloud of interplanetary dust at a constant velocity  $v_0$ . The dust particles have negligible mass. Depending on its initial position when the planet is still far away, each dust particle will either hit or miss the planet as it moves by. When they strike the planet, they stick. The capture cross section  $\sigma$  is defined as the transverse area within which all dust particles are captured. Compute  $\sigma$ . *Hint:* it is useful to consider the capture process in the reference frame of the planet.



## J05M.3 - Waves in a Series of Pendula

### Problem

Consider wave propagation in a one-dimensional medium which consists of a large number of pendula of mass  $m$  and length  $l$  coupled by springs of spring constant  $K$ . The distance between adjacent masses is  $a_0$ , which is also the natural length of the springs.

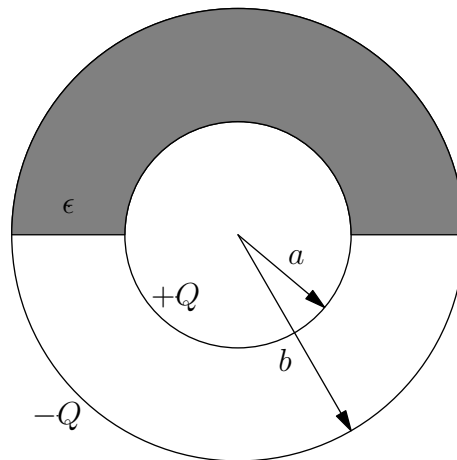


- Write the equation of motion for small horizontal displacements of the  $n$ -th mass,  $\Psi_n$ .
- Derive a dispersion relation for the propagating modes.
- What is the range of frequencies (bandwidth) over which waves can propagate along the chain?

## J05E.1 - Spherical Half-Filled Capacitor

### Problem

Two concentric conducting spheres of radii  $a$  and  $b$  carry charges  $+Q$  and  $-Q$  as shown. The radial gap between the spheres is half filled with a material of dielectric constant  $\epsilon$  and half filled with vacuum.



- Find the electric field  $\vec{E}$  and the displacement field  $\vec{D}$  everywhere between the spheres.
- What is the bound charge density on the surfaces of the dielectric?

## J05E.2 - Light Incident on a Medium

### Problem

An electromagnetic wave of frequency  $\omega$  propagates through vacuum along the  $z$  axis and is incident on homogeneous medium which fills space for  $z \geq 0$ . The medium has a magnetic permeability  $\mu$  and real dielectric constant  $\epsilon$ . The medium has a large conductivity  $\sigma$  that is a known real function of the frequency  $\omega$ . Within the medium, the  $z$  dependence of the electric field amplitude is:

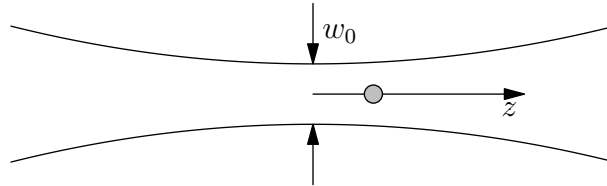
$$\vec{E} = \frac{1}{2} \{ \vec{E}_0 e^{i(kz - \omega t)} + \vec{E}_0^* e^{-i(k^* z - \omega t)} \}$$

- Obtain an expression for the complex propagation constant  $k$  in terms of  $\epsilon, \omega$  and  $\sigma$ .
- Calculate the phase of the reflected wave, relative to that of the incident wave, at  $z = 0$  in terms of  $\epsilon, \omega$  and  $\sigma$ .
- A conductor with  $\sigma = 10^{16} \text{ s}^{-1}$  ( $\sigma \simeq 10^6 \text{ Ohm}^{-1} \text{ m}^{-1}$  in SI units) reflects 90% of the incident radiation. Assume the  $\epsilon$  and  $\mu$  are the corresponding values in vacuum. What is the frequency  $\omega$  of the incident wave?

## J05E.3 - Dielectric Sphere in a Laser Beam

### Problem

A laser beam can be used to trap a dielectric object. Let the dielectric object have a radius  $a$  and be immersed in an electric field,  $\vec{E}$ . The induced dipole moment is  $\vec{p} = -\alpha\alpha^3\vec{E}$ , where  $\alpha$  is the polarizability of the material.



- The sphere is placed in a *uniform* static electric field. What is the net force,  $\vec{F}_0$ , on the sphere?
- Now the sphere is placed in a collimated (zero divergence) laser beam of power  $I$ , diameter  $w_0$ , and wavelength  $\lambda$ . Assume  $w_0, \lambda \gg a$ . What is the net force,  $\vec{F}_s$  on the sphere?
- Now the sphere is placed in a focused laser beam, so that the intensity as a function of longitudinal distance  $I(z)$ , is

$$I(z) = \left[ \frac{P}{\pi w_0^2} \right] \frac{1}{1 + \left(\frac{z}{w_0}\right)^2}$$

What is the stable position,  $z_s$ , of a particle in this beam? Assume  $z_s \ll w_0$ .

## J05Q.1 - Fermion Entanglement

### Problem

Two spin 1/2 particles interact via the Hamiltonian

$$H = -J\vec{S}_A \cdot \vec{S}_B.$$

At time  $t = 0$ , spin  $A$  points in the positive  $z$ -direction and spin  $B$  points in the negative  $Z$ -direction. Compute the density matrix of spin  $A$  at time  $t$ . At which time does it describe a pure state, that is, at which time does the entanglement between the two spins vanish?

## J05Q.2 - Charged Particle on a Ring

### Problem

A particle of charge  $e$  is free to move on a circular ring of radius  $R$  centered around a fixed particle of charge  $+e$ . The ring is in the  $x$ - $y$  plane. A uniform electric field  $E$  is applied in the  $x$ -direction.

- a) Compute the ground state energy to leading order in small  $E$ .
- b) Develop an approximation for the ground state energy in the limit of large  $E$ .



### J05Q.3 - Heavy Particle Passing a Hydrogen Atom

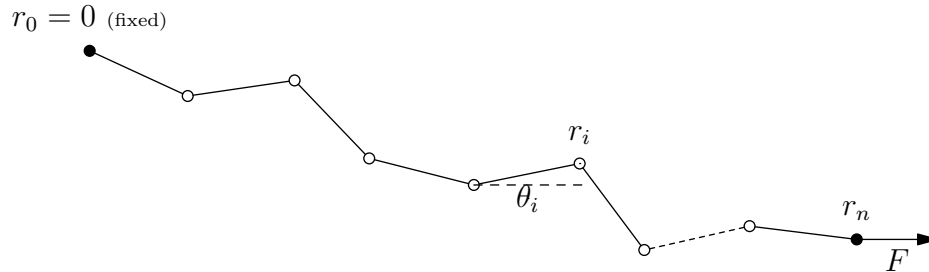
#### Problem

A heavy particle has a charge  $e$  and travels with velocity  $v$  on a straight trajectory with minimal distance  $D$  from the nucleus of a hydrogen atom (which you may assume to be fixed). Assume that initially the hydrogen atom is in its ground state. Moreover,  $D \gg a$  (where  $a$  is the Bohr radius), and  $v \gg D|E_0|/\hbar$  (with  $E_0$  the ground state energy of the hydrogen atom). What is the probability that the electron in the hydrogen atom is in a  $2p$  state after the passage of the heavy particle?

## J05T.1 - Polymer Chain

### Problem

A simple model of a polymer consists of  $N + 1$  monomers connected by bonds of fixed length  $a$ . Let the chain be confined to a plane, so that the position of each monomer  $i$  is a vector  $\mathbf{r}_i$  in two dimensions; alternatively the polymer can be described by angles  $\theta_i$  that measure the orientation of the bond from monomer  $i - 1$  to monomer  $i$ . Let the monomer  $i = 0$  be fixed at  $\mathbf{r}_0 = 0$ .



- Assume that all allowed configurations of the chain have equal energy, and that there are no externally applied forces. For  $N \gg 1$ , what is the mean end-to-end length of the chain? What (Gaussian) probability distribution  $P(\mathbf{r}_N)$  of the position  $\mathbf{r}_N$  of the free end of the polymer does this imply?
- A force  $\mathbf{F}$  is applied to the free end of the polymer. How does the (analog of the ) Gibbs free energy  $G(T, \mathbf{F}, N)$  of the polymer depend on  $|\mathbf{F}|$ ? (Give the leading term in an expansion of  $G(T, \mathbf{F}, N) - G(T, 0, N)$  in powers of  $\mathbf{F}$  for  $N \gg 1$ .)
- Obtain the corresponding expression for the (analog of the) Helmholtz free energy  $A(T, \mathbf{r}_N, N)$ . How do the internal energy  $U$  and entropy  $S$  depend on  $\mathbf{F}$ ?
- Show that for small forces the polymer acts like a Hooke's law spring, and compute the corresponding stiffness. When this "spring" is held stretched, where is the energy stored? (Describe the nature of this stored energy.)

## J05T.2 - Bose Einstein Condensation

### Problem

Consider  $N \gg 1$  spinless noninteracting bosons contained in a n isotropic three-dimensional harmonic well. In terms of the position  $\vec{r}$  and the momentum  $\vec{p}$ , the single-particle Hamiltonian is

$$H = \frac{1}{2m}|\vec{p}|^2 + \frac{1}{2}m\omega_0|\vec{r}|^2,$$

where the particles have mass  $m$  and the oscillations in the potential have natural frequency  $\omega_0$ . The resulting energy levels depend on the three quantum numbers

$$E = \hbar\omega_0(n_x + n_y + n_z + (3/2)),$$

where each  $n_i = 0, 1, 2, \dots$ . This can also be represented as energy levels that depend on a single quantum number  $n = 0, 1, 2, \dots$ ;  $\epsilon_n = \epsilon_0 + n\hbar\omega_0$ , but with a degeneracy  $g_n = (n+1)(n+2)/2$ , and  $\epsilon_0 = \frac{3}{2}\hbar\omega_0$ .

- What is the specific heat  $c_N(T)$  per particle, at fixed particle number  $N$ , in the “classical limit” where  $k_B T / \hbar\omega_0$  is so large that  $N_0 \ll 1$ , where  $N_0$  is the mean number of particles in the  $n = 0$  state.
- Find  $c_N(T)$  at low temperatures  $k_B T \ll \hbar\omega_0$ , including the leading behavior for nonzero temperatures.
- Find the chemical potential,  $\mu(T, N)$  in the “classical limit”. Above what temperature scale is the “classical limit” reached?
- Find  $\mu(T, N)$  for low temperatures  $k_B T \ll \hbar\omega_0$ , including the leading behavior for non-zero temperatures. (Hint: find an exact expression for  $\mu(T, N_0)$  at all temperatures and substitute the value of  $N_0(T, N)$  in the temperature range of interest.)
- Since the particles are bosons,  $N_0(T, N)$  may be macroscopic (i.e. of order  $N$ ) in a finite temperature range  $T < T_{BEC}(N, \hbar\omega_0)$ . Obtain an expression for  $T_{BEC}$  in the large- $N$  limit. You can express any numerical constants as dimensionless integrals which you must define, but need not evaluate. (Hint: for  $N \gg 1$ ,  $k_B T_{BEC} \gg \hbar\omega_0$ .)

*Note: For large but finite  $N$ , there is no true phase transition at  $T = T_{BEC}$ , but a qualitative change in the system takes place over a small temperature range  $\Delta T$  around  $T_{BEC}$  where  $\Delta T / T_{BEC} \ll 1$ .*

## J05T.3 - Thermodynamic Variables

### Problem

A thermodynamic system has the following relation between its entropy  $S$ , volume  $V$ , internal energy  $U$ , and particle number  $N$ :

$$S(U, V, N) = \gamma(UVN)^{1/3},$$

where  $\gamma$  is a constant.

- a) Derive a relation connecting  $U, N, V$  and the temperature  $T$ .
- b) Find the heat capacity  $C_{VN}$  at constant  $V$  and  $N$ , as a function of  $V, N$ , and temperature  $T$ .

Now assume you are given two identical bodies with the above properties.  $N$  and  $V$  are the same for both, and are fixed, but the two bodies have different initial temperatures,  $T_1$  and  $T_2$ .

- c) If the two bodies are placed in thermal contact, and left alone until heat flow ceases and equilibrium is reached, what is their common final temperature  $T_f$ ?
- d) If the flow of heat between the bodies is used to drive an engine that does the maximum possible amount of useful work  $W_{max}$  before the two bodies reach a common final temperature  $T'_f$ , what is that temperature? What is  $W_{max}$ ?