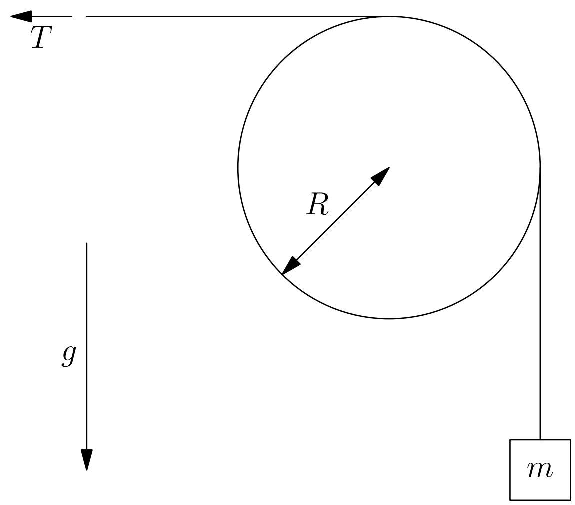
M98M.1—Mass on a Rope and Cylinder

Problem

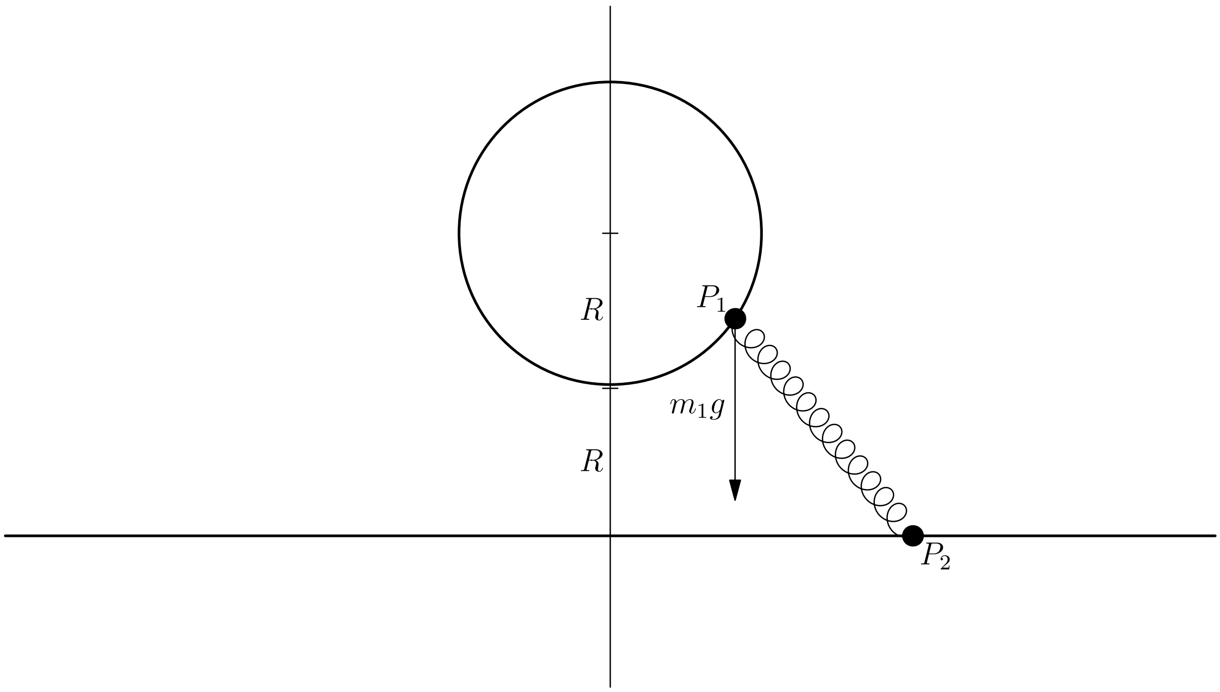
A mass m is lifted by means of a rope drawn across a cylinder as sketched in the figure. The cylinder is fixed so that it does not rotate. A steady horizontal tension T is applied, and the mass rises vertically with no acceleration. Find an expression for T in terms of the coefficient of kinetic friction, μ, between the cylinder and the rope.



M98M.2—Masses Connected by a Spring

Problem

Consider a system of two particles, each of mass m, in a constant gravitational field g. Particle P_1 moves without friction on the vertical circle of radius R. P_2 moves without friction along the horizontal line. The two particles are connected by a perfect spring whose elastic constant is k. The spring is prestressed so that the tension is proportional to the length, $T = kr$, when the spring length is r.



a) What are the position(s) of equilibrium? Specify for each whether the position is stable or unstable.

b) For each of the stable position(s), and for each normal mode of small oscillations, sketch the motions of the particles.

c) Find the frequencies of the normal modes of small oscillations around the stable positions.

M98M.3—Fluid Dynamics

Problem

Answer all FIVE of the following short questions. Where a numerical answer is required use acceleration of gravity $g = 980$ cm s^{-2}, atmospheric pressure $p = 1 \times 10^6$ dynes cm^{-2} $= 1 \times 10^5$ N m^{-2}, and density of water $\rho = 1$ g cm^{-3}.

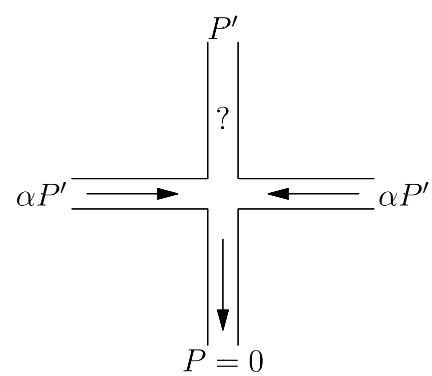
a) Water is flowing out of a hole with diameter $d = 1$ cm in the vertical side of a container. The center of the hole is $h = 1$ m below the top of the water. The diameter of the container is $D = 10$ m. Compute the speed of the water as it passes through the hole under the assumptions that the flow is laminar, and that viscous drag is negligible.

b) A balloon filled with helium (density $\rho_{He} = 2 \times 10^{-4}$ g cm^{-3}) is tied to the floor of a train car by a string of length $L = 1$ m. The car is accelerating forward on a level surface with acceleration $a = 1$ m s^{-2}. Sketch the position of the balloon in the train car and give an expression for the angle the balloon's string makes with the vertical.

c) An individual is standing on a level piston that can move freely in a cylinder. The piston is supported by water in the cylinder beneath it. The water is connected to a pipe that rises vertically beside the person. The cylinder has a circular cross section with radius $R = 1$ m. The pipe has a circular cross section with radius $r = 1$ cm. The individual and piston have total mass $M = 100$ kg. Find the difference between the heights of water in the cylinder and in the pipe.

d) A spherical soap bubble of radius $r = 1$ cm is blown from soap which has surface tension $\gamma = 50$ dynes cm^{-1} $= 0.05$ N m^{-1}. What is ΔP, the pressure difference between the inside of the bubble and the outside?

e) Four horizontal cylindrical tubes intersect as shown in the figure. The tubes have equal lengths L and radii R, with $L \gg R$. A fluid of viscosity η flows laminarly in the tubes. The ends of two opposing tubes are held at pressure $\alpha P'$, while the end of the third arm is maintained at (approximately) zero pressure. The end of the fourth tube is at pressure P'. For what values of α will the flux of fluid in the fourth tube be outward from the junction?



M98E.1—Field of a Wire

Problem

A neutral wire along the z-axis carries current I that varies with time t according to

$$I(t) = \begin{cases} 0 & t \leq 0, \\ \alpha t & t > 0, \end{cases} \qquad \alpha \text{ is a constant.}$$

Deduce the time-dependence of the electric and magnetic fields, **E** and **B**, observed at a point $(r, \theta = 0, z = 0)$ in a cylindrical coordinate system about the wire. Use your expressions to discuss the fields in the two limiting cases that $ct \gg r$ and $ct = r + \epsilon$, where c is the speed of light and $\epsilon \ll r$.

Fact:

$$\int \frac{dx}{\sqrt{a^2 + x^2}} = \ln(x + \sqrt{a^2 + x^2}).$$

M98E.2—Cylindrical Magnet and a Steel Sheet

Problem

A cylindrical magnet has a cross-section area A, length L and uniform magnetization M parallel to L. The magnet is very long, $L \gg A^{1/2}$. It is placed on end against a steel sheet, with the axis of the cylinder perpendicular to the surface of the steel sheet. The steel has infinite magnetic permeability. What force F is necessary to pull the magnet from the sheet? Neglect gravity.

M98E.3—Magnetic Field of the Earth

Problem

The magnetic field of the Earth may be approximated as a magnetic dipole. The axis of the dipole does not coincide with the geographic North pole, but is inclined at angle $\Psi = 11°$.

a) Estimate the magnetic moment of the Earth, using the fact that the magnetic field is about 0.5 gauss at the equator.

b) Find the radiated power in Watts due to the Earth's rotation under the assumption that the Earth is isolated in empty space.

c) The Earth is actually immersed in the the solar wind. The density of the solar wind n_e is roughly 10 proton/cm^3. Explain whether the radiation computed in part b) is detectable outside the solar system.

M98Q.1—Two Interacting Particles

Problem

A system of two particles, one with spin s_a, the other with spin s_b, is governed by the Hamiltonian

$$H = K + V(\vec{r}_a, \vec{r}_b) + f(\vec{r}_a, \vec{r}_b)(S_3^{(a)} - S_3^{(b)}).$$

Here K is the kinetic energy operator and $\vec{S}^{(a)}$ and $\vec{S}^{(b)}$ are the spin operators. Let $\vec{S} = \vec{S}^{(a)} + \vec{S}^{(b)}$ be the total spin operator. You are not given the functions V and f, but you *are* told that $f(\vec{r}_a, \vec{r}_b) < 0$. The potential V is sufficiently attractive so that there exists at least one bound state.

a) Work out the (unique) ground state expectation value of $\vec{S}^2$, the square of the total spin angular momentum.

b) Now take the special case $s_a = 1, s_b = 1/2$. What are the possible outcomes of a measurement of $\vec{S}^2$? For the ground state, what are the probabilities of these outcomes?

M98Q.2—Scattering From a Spherical Potential

Problem

a) Calculate the differential cross-section, $d\sigma/d\Omega$, for a particle with mass m in the spherical potential $V(r) = V_0\, e^{-(r/a)^2}$, in first-order Born approximation. You may need

$$\int_0^\infty \sin(r) e^{-(r/b)^2} r dr = \frac{\sqrt{\pi}}{4} b^3 e^{-b^2/4}.$$

b) Calculate the total cross-section. It may be helpful to use the representation $|\vec{k} - \vec{k}'| = 2|\vec{k}| \sin(\theta/2)$, where θ is the angle between $\vec{k}$ and $\vec{k}'$.

c) For which values of V_0, a and/or k is the first-order Born approximation applicable?

M98Q.3—Variational Principle

Problem

a) Let $|\lambda\rangle$ be the ground state and $E(\lambda)$ the ground state energy of a Hamiltonian $H(\lambda) = H_1 + \lambda H_2$ that depends linearly on a real parameter λ. Use the variational principle based on the trial wave function $|\tilde{\lambda}\rangle$ with $\tilde{\lambda}$ fixed to show that $E(\lambda)$ is a concave function ($E''(\lambda) \leq 0$) in λ.

[Hint: It is enough to show that $E(\lambda) - E(\tilde{\lambda}) \leq (\lambda - \tilde{\lambda})\langle\tilde{\lambda}|H_2|\tilde{\lambda}\rangle = (\lambda - \tilde{\lambda})E'(\tilde{\lambda})$.]

b) Consider the Hamilton operator given by the following matrix,

$$H(a,b) = \begin{pmatrix} 1 & a & ab \\ a & 1 & b \\ ab & b & 1 \end{pmatrix}, \qquad b \neq 0,$$

depending on real parameters a and b. Let $E(a,b)$ be the corresponding ground state energy. Show that the function $E(a,b)$ is concave in each of the parameters.

c) Show that $E(a,b)$ in part b) is monotone decreasing in a for $a \geq 0$ and fixed $b \neq 0$.

[Hint: Start by showing that $\left.\frac{\partial E(a,b)}{\partial a}\right|_{a=0} = 0$.]

M98T.1—Carnot Engine

Problem

A Carnot engine uses n moles of an ideal gas as its working substance. The absolute temperatures of its hot and cold reservoirs are denoted by T_1 and T_2, respectively. The net work performed by the engine in one cycle of operation is W. The specific heats of the gas may be assumed independent of the temperature. An investigator is asked to check the values of the reservoir temperatures, but unfortunately she is not provided with a thermometer. However, she is able to measure W, and also the following volumes:

V_1 = volume of working substance when first contacted with hot reservoir,
V_2 = volume of working substance after extracting heat from hot reservoir,
V_3 = volume of working substance when first contacted with cold reservoir,
V_4 = volume of working substance after giving up heat to cold reservoir.

Derive expressions for the unknown temperatures, T_1 and T_2, in terms of n, W, ratios of the above volumes, the molar gas constant R, and the ratio γ of the constant pressure and constant volume specific heats for the gas.

M98T.2—Helium Atoms in a Box

Problem

A cube of volume $V = L^3$ contains 10^{22} freely moving ^{4}He atoms. What fraction of particles is in the first excited level at temperature $T = 10^{-3}$ K, when $L = 1$ cm? The mass of each atom is 6.6×10^{-24} g, $h = 6.6 \times 10^{-27}$ erg s, and $k_B = 1.38 \times 10^{-16}$ erg/K.

M98T.3—Anharmonic Oscillator

Problem

Using the anharmonic potential $V(x) = cx^2 + gx^3 + fx^4$, for a one-dimensional classical harmonic oscillator, find the approximate heat capacity at low temperatures including terms of order T.