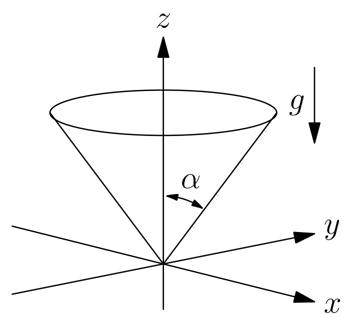
M02M.1—Particle in a Cone

Problem

A small particle of mass m is constrained to slide, without friction, on the inside of a circular cone whose vertex is at the origin and whose axis is along the z-axis. The half angle at the apex of the cone is α and there is a uniform gravitational field g, directed downward and parallel to the axis of the cone.



a) Determine a set of generalized coordinates, and obtain the equations of motion in these coordinates.

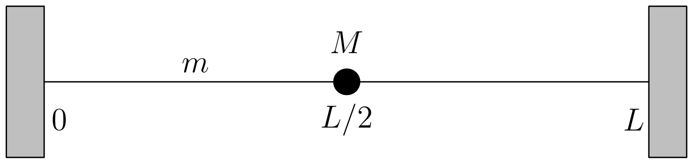
b) Show that a solution of the equations of motion is a circular orbit at a fixed height z_0. Obtain an expression for the frequency, ω, of this motion.

c) Show that the circular motion is stable. If Ω is the frequency of small oscillations about the unperturbed motion, show that the ratio Ω/ω depends only on α. Determine this dependence.

M02M.2—Mass on a String

Problem

A string of length L, mass m, and tension τ is fixed at the two extremes. A mass M is attached in the middle, as shown in the figure. The string oscillates transversely in the plane of the figure.



a) Write down the form of the normal modes and the equations that determine their frequencies.

b) Show that the equations you derived have a family of solutions where the mass M is always at rest.

c) List the normal mode frequencies at $M = 0$ and $M = \infty$.

d) Sketch how the first four frequencies interpolate between these limits.

M02M.3—The Coriolis Effect

Problem

A particle is dropped vertically in the Earth's gravitational field at latitude λ. Assume it feels an air drag $F = kv^2$. Due to the Coriolis effect, it will undergo a horizontal deflection.

a) Initially neglect the Earth's rotation. Find an explicit equation for the vertical velocity.

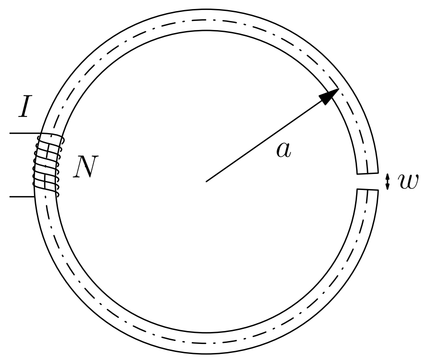
b) Working at leading order in the Earth's angular velocity ω, and using the result you just derived, find the horizontal velocity as a function of time.

c) What is the velocity at $t \gg \sqrt{\frac{m}{gk}}$?

M02E.1—Iron Ring with a Gap

Problem

N turns of a wire are wrapped around an iron ring in which a small gap has been cut. The radius of the ring is a and the width of the gap is w, with $w \ll a$. A current I flows in the wire. The magnetic permeability of the iron is μ.



a) Find the $\boldsymbol{B}$ field in the gap.

b) Find the force per unit area on the faces of the gap. Does the gap have the tendency to widen or contract?

M02E.2—Dielectric Cylinder in an Electric Field

Problem

A cylinder of radius a and dielectric constant ϵ is placed along the z-axis in a electric field, whose form is $\boldsymbol{E}_i = E_0\hat{\boldsymbol{x}} + E_1[(x/a)\hat{\boldsymbol{x}} - (y/a)\hat{\boldsymbol{y}}]$ *before* the cylinder is placed in the field. Give expressions for the total electric field $\boldsymbol{E}$, the displacement field $\boldsymbol{D}$ and the polarization density $\boldsymbol{P}$ everywhere.

M02E.3—Current in a Cylindrical Wire

Problem

A current I, carried by freely moving electrons, runs through a cylindrical wire with radius r_0. Assuming that the electrons are moving with velocity v and that the total charge per unit length in the wire vanishes (in the lab frame), find the radial profile of the current. What is the voltage difference between the center and the edge of the cylinder?

M02Q.1—Driven Harmonic Oscillator

Problem

Consider the driven harmonic oscillator:

$$H(t) = \frac{1}{2}(p^2 + x^2) - \sqrt{2} f(t) x,$$

where $f(t)$ is a c-number function of time. Note that we have set the natural frequency of the oscillator ω_0, to 1. In the following you should also use $\hbar = 1$.

Defining the time evolution operator in the Schrödinger picture by,

$$|\psi(t)\rangle_S = U(t)|\psi(0)\rangle$$

we can transform to the Heisenberg picture

$$|\psi\rangle_H = U^\dagger(t)|\psi(t)\rangle_S \equiv |\psi(0)\rangle$$

and

$$O_H(t) = U^\dagger(t) O_S U(t).$$

Working in the Heisenberg picture,

a) Write down the equations of motion for the operators x and p.

b) Solve the operator equations of motion derived in a) for the case

$$f(t) = \begin{cases} f_0 \cos \omega t & \text{for } 0 \leq t \leq T \\ 0 & \text{otherwise} \end{cases}$$

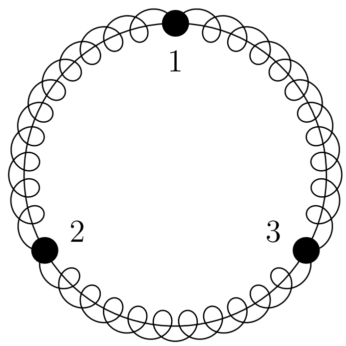
c) Compute the expectation value of the total energy gained by the oscillator at resonance, $\omega = 1$, if $|\psi(0)\rangle = |0\rangle$, its unperturbed ground state. Sketch your result as a function of T.

(You may wish to work with creation/annihilation operators to make the algebra more familiar.)

M02Q.2—Three Particles on a Ring

Problem

Consider a molecule of three particles, each carrying $S = \frac{1}{2}$ connected by springs and constrained to move on a circle of circumference L. (The last bit is to make the geometry easier, the situation without this constraint works out the same way.)



The Hamiltonian for this system is

$$H = 2J \sum_{i=1}^{3} (1 - \alpha x_{i,i+1}) \boldsymbol{S}_i \cdot \boldsymbol{S}_{i+1} + \frac{1}{2} k x_{i,i+1}^2$$

where the $\boldsymbol{S}_i$ are spin-1/2 operators and the $x_{i,i+1}$ (> 0) are the distances along the circle between neighboring particles i and $i+1$. By periodicity, $4 \equiv 1$. Finally, $J > 0$.

a) Find the ground states of the spin system in isolation, i.e. when $\alpha = 0$. How many ground states are there?

b) Find the ground states of the coupled system when $0 < \alpha \ll 1$. Restrict yourself to ground states in which two of the three interparticle distances are identical, e.g. $x_{1,2} = x_{2,3}$.

(You may find the following identity useful in the computations in part b):

$$\boldsymbol{S}_i \cdot \boldsymbol{S}_j = \frac{1}{4}(2P_{ij} - 1)$$

where P_{ij} interchanges spins i and j.)

M02Q.3—Scattering From a Magnetic Barrier

Problem

Consider a charged particle moving in the x-y plane subject to a perpendicular magnetic field $B_z = B_0\theta(x)\theta(d-x)$. The magnetic field is constant in a strip of width d and zero everywhere else. We will study the problem of scattering of plane waves from this "magnetic barrier".

a) Write down the Schrödinger Hamiltonian for this problem. You have to choose a gauge for the vector potential - choose the gauge $A_x = A_z = 0$, and also choose $A_y = 0$ for $x < 0$.

Consider the scattering problem for an electron incident from $x < 0$ and moving perpendicular to the barrier. For an incident wave $\exp(ikx)$ there will, in general, be a transmitted wave $T\exp(i\tilde{k}x)$ and a reflected wave $R\exp(-ikx)$.

b) The transmitted wave vector $\tilde{k}$ is determined by simple kinematics in terms of k and $B_0 d$. What is this relation?

c) For a given barrier, you will find that, below a certain critical energy E_0, $\tilde{k}$ is imaginary. What does this mean? Give a classical argument that leads to the same critical energy.

d) What is the direction of the transmitted probability flux? It is <u>not</u> along the x-axis!

e) Find the reflection and transmission coefficients in the limit $d \to 0$, with $B_0 d$ fixed.

M02T.1—Dilute Gas in Gravity

Problem

Consider a vertical container of dilute gas in a constant gravitational field. The bottom of the container is kept at temperature T; otherwise it is thermally isolated.

a) How does the density ρ vary as a function of height, h?

b) Under the assumptions used to derive this variation, what is the distribution vertical momenta $F(p)$ of the gas atoms at height h?

c) Assuming we have a (strictly) ideal gas of noninteracting particles should we expect the conclusion of part b) to hold?

d) Assume the situation in c) and assume that the gas at the bottom has the momentum distribution characteristic of temperature T. What is the distribution of vertical momenta of the gas atoms at height h?

M02T.2—Hydrogen Formation

Problem

In this problem we investigate the formation of hydrogen atoms in the early universe. Although the binding energy of hydrogen is 13.6 eV, the majority of protons and electrons did not become bound into atoms until the temperature of the neutral primordial plasma cooled to about 0.3 eV.

In the following we make four assumptions:

1. The hydrogen atom has no bound states apart from its ground state.
2. We ignore other bound complexes that might be formed, e.g., hydrogen ions and molecules.
3. All interactions among hydrogen atoms, protons and free electrons are ignored (apart from the fundamental process of atom formation).
4. Everything is in thermal equilibrium.

There are two questions:

a) Assume that at $T = 0.3$ eV half of the protons had a bound electron. From this information calculate the densities (in units of particles per cubic centimeter) of free electrons, free protons, and hydrogen atoms.

b) At $T = 0.3$ eV what is the density of photons? Is it much smaller than, comparable to, or much larger than the densities obtained in part a)?

M02T.3—Pauling's Ice Rule

Problem

One of the all-time classic experiments was the measurement of the residual entropy S of ordinary water-ice by Giauque and Stout. This is the entropy that ice has at a temperature of zero kelvin. This entropy is not zero.

a) What 'law' does this violate? How would you make such an absolute determination of this entropy?

Pauling proposed a model to explain this entropy and the problem here will be to calculate S according to this model. The crystal structure of ice is such that each oxygen atom has 4 nearest neighbor oxygen atoms. On each 'bond' between nearest neighbor oxygen pairs sits a hydrogen atom, but it does not sit in the middle. It sits in one of two positions close to one of the two oxygens at the end of the bond.

b) Assume there are N water molecules. If all hydrogen configurations are equally likely, what is S?

Not all configurations are equally likely. Pauling introduced his "ice rule": There are precisely two hydrogen atoms close to each oxygen atom, reflecting the molecular structure of water. Otherwise, all configurations are equally likely. This limits the number of configurations, but presents a horrific combinatorial problem. Pauling simplified matters further with his Pauling approximation: He estimated the fraction of allowed configurations by taking the hydrogen configuration surrounding one oxygen atom to be independent of those surrounding all other oxygen atoms.

c) Compute S on the basis of the Pauling approximation.

(This answer agrees well with the experimental number $0.44k_B$ per molecule. The original Pauling paper is J. Am. Chem. Soc. **57**, 2680 (1935).)