Section A. Mechanics

1. **Coriolis Effect.** A particle of mass m is launched from the Earth's surface at colatitude θ with initial velocity v_0 straight up. A drag force $\vec{F} = -b\vec{v}$ acts on the particle while in flight, where $b = mg/v_0$. You may take the acceleration of gravity to be constant throughout the motion.

   Use a coordinate system with $\hat{\imath}$ pointing East, $\hat{\jmath}$ pointing North, and $\hat{k}$ straight up, so that the initial conditions are $x(0) = y(0) = z(0) = 0$, $\dot{x}(0) = \dot{y}(0) = 0$, and $\dot{z}(0) = v_0$.

   a) Ignoring the Coriolis force, what is the vertical velocity $\dot{z}(t)$?

   b) Now taking into account the Coriolis force and working at leading order in the Earth's angular velocity $\vec{\omega}$, what are the horizontal components of the velocity, $\dot{x}(t)$ and $\dot{y}(t)$?

   c) Relative to its launch position, where does the particle land? You may assume that v_0 is such that the particle reaches terminal velocity on the way down.

2. **Higgs Orbits**. A particle of mass m moves under the influence of a potential

$$V(r) = -ar^2 + br^4, \tag{1}$$

where a and b are positive constants and r is the distance between the particle and the force center. In the context of quantum field theory with r being a complex scalar field, this is the so-called "Higgs Potential." Here we interpret the potential as arising from a central force and investigate the possible motion of a particle in this field.

a) What is the radius ρ of the circular orbit allowed in this potential?

b) What is the condition on a and b such that this orbit is stable?

c) What is the frequency of small oscillations around $r = \rho$?

In the Higgs analogy, the answer to part a corresponds to the vacuum expectation value of the field, while the answer to part c corresponds to the mass of the Higgs boson. The stability of the vacuum (part b) was used as early as 1976 to set a lower bound on the mass of the Higgs boson.

3. **Precession of Equinoxes**. The Sun and Moon exert torques on the Earth that cause its rotation axis to precess around the normal to the plane of the ecliptic once every 26000 years. The component of the gravitional potential from the Sun that gives rise to a torque on the Earth is:

$$V = \frac{GM\left(I_3 - I_1\right)}{2r^3}\left[\frac{1}{2} - \frac{3}{2}\cos^2\theta\right], \tag{2}$$

where M is the mass of the Sun, r is the radius of the Earth's orbit, I_1 and I_3 are moments of inertia around Earth's principle axes, and θ is the tilt of the Earth's rotation axis relative to the normal to the plane of the ecliptic. For orientation, the angular frequency of rotation of the Earth is ω_3, and the potential has been averaged over one orbital period since the precession period is much longer than a year.

a) Write down the Lagrangian describing the Earth's rotational and orbital motion in the graviational field of the Sun assuming no nutation of the rotation axis ($\dot{\theta} = \ddot{\theta} = 0$).

b) Assuming the precession frequency $\dot{\phi}$ is much less than the rotational frequency ω_3 of the Earth ($\dot{\phi} << \omega_3$), derive a formula for the ratio $\dot{\phi}/\omega_0$, where ω_0 is the angular frequency of Earth's orbital motion. For full credit your answer should be in terms of ω_0, ω_3, I_1, I_3, and θ only.

c) Looking down on the northern hemisphere from above the plane of the ecliptic, in which direction does the Earth's rotation axis precess?

d) Using $\left(I_3 - I_1\right)/I_3 = 0.003$, $\theta = 23.5°$, and the known frequencies of Earth's rotation and orbit, give a rough estimate for the precession period in years. [Note that you should obtain a value larger than 26000 years since the torque from the Moon is larger than, and adds to, that from the Sun.]

Section B. Electricity and Magnetism

1. **Field Transformations**

(a) Let a current I circulate in a square of wire of side d lying in the x-y plane, with center at the origin. What is the vector potential $\vec{A}$ at a position x_o, where $x_o >> d$ (ie, Taylor expand the denominator)? (b) What is the magnetic field B at x_o?.

(c) At x_o lies a charge q_o at rest. Calculate the force acting on the charge, and the force actiing on the loop.

(d) Now boost to a frame where the charge q_o and the loop are both moving with speed $+v_o\hat{x}$. What is the electric field $\vec{E}'$ due to the loop acting on the charge (I need magnitude and direction). For this, you don't need the answer for part (b), just call the field B but you need the direction.

(e) What is the total force $\vec{F}_{tot}$ acting on the charge q_o in this frame?

2. **Radiating Fields**

(a) A hydrogen atom has a diameter D of about 1×10^{-8} cm. What is the frequency ω_0 of the rotation of the electron around the much more massive proton? Ignore relativity.

(b) Replace the rotating electron with a negative charge oscillating back and forth with the angular frequency ω_o. Find the average power I_{tot} radiated over all space. You will need to do the power integral to get full credit, starting from the radiated field at position r and angle θ.

(c) Make the incorrect assumption that the average radiated power is constant with time and estimate the lifetime of a hydrogen atom in seconds.

3.

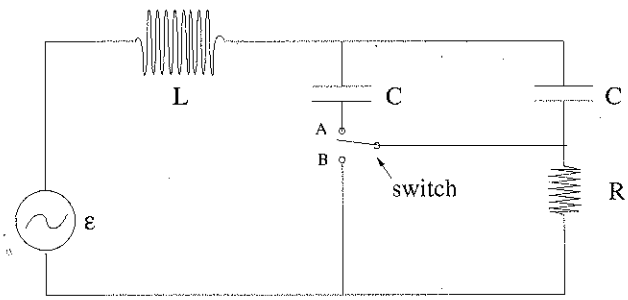

Figure 1: An AC circuit.

Complex Impedances

(a) Consider the circuit above. The switch can be set in any of three positions, A, B or open (unconnected). The source supplies a voltage $\varepsilon(\omega) = \varepsilon_0 e^{i\omega t}$.

When the switch is connected to A, find the frequency ω that maximizes the current through the resistor R. (5 pts)

(b) If we then flip the switch to the B position, what is the average power dissipation in the circuit (ignoring transient effects).

C) We now open the switch to the middle position. Find the value of the resistor R that will drop the amplitude of the current to 1/2 the value you found in part a), at the same frequency ω you found in part a). (5 pts)

(D) Suppose that the inductor, of inductance L, is constructed from a solenoid with N turns over a length ℓ, whose axis of symmetry lies on the $\hat{x}$ axis.

Express the cross sectional area of the solenoid in terms of the inductance L, the number of turns N, the length ℓ and any fundamental constants.

Section C. Quantum Mechanics

1. A quantum particle moves in one dimension with energy as a function of wavenumber $E(k)$. Its momentum is $p = \hbar k$ and is conserved. At time $t = 0$ the wavefunction $\psi(x, t = 0)$ of this particle is a minimum-uncertainty wavepacket centered at the origin ($x = 0$) in real space and with average momentum $\langle p \rangle_{t=0} = \hbar k_0$. Assume that the initial uncertainty in the position $\sqrt{\langle x^2 \rangle_{t=0}} = \sigma$ is large but finite, so the uncertainty in the momentum is small but nonzero. Thus approximate $E(k)$ by its Taylor expansion about k_0 keeping terms only to order $(k - k_0)^2$.
(a) In terms of the given parameters; $E(k_0)$; and $\frac{dE}{dk}$ and $\frac{d^2E}{dk^2}$ evaluated at $k = k_0$, obtain the normalized wavefunction $\psi(x, t)$ at nonzero times t. Do not make any assumption about the dispersion relation $E(k)$ other than that its first and second derivatives exist and are finite at k_0.

(b) Calculate the expectation values: $\langle x \rangle_t$, $\langle p \rangle_t$, $\langle (x - \langle x \rangle_t)^2 \rangle_t$ at nonzero times t.
[If you get bogged down: first do this problem assuming $\frac{d^2E}{dk^2} = 0$ before letting it be nonzero.]

2. Consider two indistinguishable nonrelativistic **bosons** of mass m, constrained to move one-dimensionally around a circle of perimeter L. The particles each have **spin-1**, and they interact via a spin-independent potential that is a Dirac delta-function: $V(x_1, x_2) = g\delta(x_1 - x_2)$, where x_i is the position on the circle (in arc length) of particle i.

(a) First look at zero interaction, $g = 0$, being careful to only include states of the correct symmetry for these indistinguishable spin-1 bosons. What are the energies and the degeneracies of the ground state and of the lowest-energy excited state? In each case, say what value(s) of total spin these states may have.

(b) Add a weak interaction $g \neq 0$. Now what are the degeneracies of the ground state and of the lowest-energy excited state? For each sign of g, say what value(s) of total spin these states may have.

(c) Solve for a two-particle ground state wavefunction, including showing the spin state. Do this first at $g = 0$, and then all other $g \neq 0$. In the latter case you may leave one parameter in the wavefunction specified only as the solution to an equation that you will not be able to solve analytically.

3. Two distinguishable but equal-mass particles move and interact in three dimensions ($\vec{r}_i = (x_i, y_i, z_i)$) with the Hamiltonian

$$H = -\frac{\hbar^2}{2m}(|\vec{\nabla}_1|^2 + |\vec{\nabla}_2|^2) + \frac{k}{2}(|\vec{r}_1|^2 + |\vec{r}_2|^2) + g(x_1x_2 + y_1y_2 - 2z_1z_2) \ .$$

Solve for the ground state wavefunction $\psi_0(\vec{r}_1, \vec{r}_2)$ when it exists, and say for what range of g it does exist (assume both m and k are positive).

Section D. Thermo and Statistical Mechanics

1. Consider a long molecule like DNA that is made out of two polymer strands, with links connecting the monomers between one strand and the other; let the number of links be N. Now imagine that we grab the ends of the two strands and pull them apart with a force F. In order to lengthen the segment that we are pulling on, we have to break links. Each time we break a link the energy of the molecule goes up by an amount Δ, the "bond energy" of each link.
On the other hand, each time we break a link, the ends we are pulling on move apart by a distance $2l_\circ$, where $l_\circ$ is the distance between the links along one strand. Thus, the energy of the molecule with n links broken is $E(n) = n(\Delta - 2Fl_\circ)$.

(a) Find an equation that relates the mean number of broken links $\langle n \rangle$ at temperature T to the partition function Z.
(b) Define $a = \left(\frac{\Delta - 2Fl_\circ}{k_B T}\right)$. Evaluate the partition function Z. Show that in the limit of large N, the behavior of Z is very different depending on whether F is smaller or larger than a "critical" value F_c. What is value of and the physical meaning of F_c?
(c) Use your result for Z to calculate $\langle n \rangle$ in terms of a in the limit of large N.

2. A recent experiment on trapped atomic gases by the Zwierlein group at MIT reports the data shown in the figure. In this problem you will try to gain an understanding of a part of the data (the highlighted curve) which is well described by a free Fermi gas.

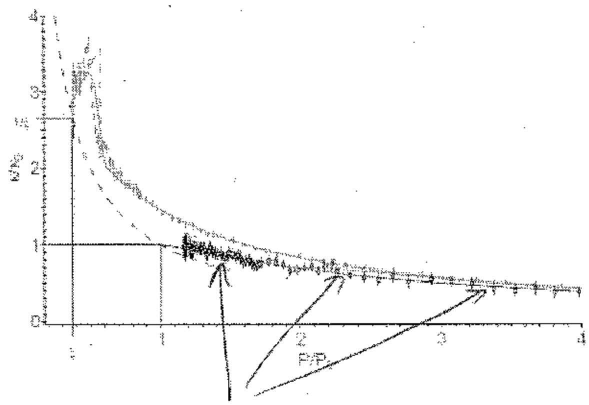

Figure 1: Normalized compressibility versus normalized pressure from Ku et al., *Science* 335, 563 (2012).

a) The isothermal compressibility of a finite volume of gas is defined as

$$\kappa = -\frac{1}{V}\left(\frac{\partial V}{\partial p}\right)_{T,N}$$

Show that in the infinite volume limit this reduces to

$$\kappa = \frac{1}{n}\left(\frac{\partial n}{\partial p}\right)_{T}$$

where $n = N/V$.

b) Consider a three dimensional gas of spinless fermions of mass m and density n at $T = 0$. Calculate its pressure $p_0(n)$ and compressibility $\kappa_0(n)$.

c) Now consider the gas at a non-zero temperature T. Let us define

$$\tilde{p}(n,T) = \frac{p(n,T)}{p_0(n)}$$

and

$$\tilde{\kappa}(n, T) = \frac{\kappa(n, T)}{\kappa_0(n)}$$

which are the quantities plotted in the figure.
Use dimensional analysis to show that $\tilde{\kappa}$ can be written as a function $\kappa(\tilde{p})$ of $\tilde{p}$ alone.

d) By construction, $\tilde{\kappa}(1) = 1$. Calculate the leading asymptotic behavior of $\tilde{\kappa} - 1$ as $\tilde{p} \to 1$. [Useful result: $p(n, T) = p_0(n)\{1 + \frac{5\pi^2}{12}(\frac{k_B T}{\epsilon_F(n)})^2 + \cdots\}$]

3. A metal has two phases, N (normal) and S (superconducting). Assume that in the normal phase the magnetization M (per unit volume) due to an applied external magnetic field H is negligible, so the magnetic induction or flux density $B \equiv \mu_0(H+M) = \mu_0 H$ in the normal metal phase.

The metal is cooled down to a temperature T in a large magnetic field H, and then H is reduced to zero. At temperatures $T < T_c$, it is observed that as H is reduced there is a critical field

$$H_c(T) = H_0 \left[1 - \left(\frac{T}{T_c}\right)^2\right],$$

where a first-order phase transition from the normal state to the superconducting state occurs. The magnetic flux is completely expelled from the metal, and $B = 0$ in its interior (Meissner effect) for $H < H_c(T)$ in the superconducting state.

Recall that the magnetic variant of the Gibbs free energy (per unit volume) is $G(T, H) = U - TS - M'H$, where $M' = \mu_0 M$ in SI units. (Ignore any thermal expansion of the metal, and treat its volume as fixed).

(a). Find the difference of the entropy densities $\Delta S(T) = S_N(T) - S_S(T)$ between the normal and the superconducting phases (with the assumption of negligible magnetization in the normal phase, $S_N(T)$ is independent of H).

(b). If the system is heated in the absence of a magnetic field (at $H = 0$) it undergoes a continuous (second-order) phase transition from superconductor to normal metal at $T = T_c$. What is the discontinuity in its specific heat per unit volume at this phase transition? (Make a sketch showing how the specific heat varies with temperature near the transition). Which phase has the larger specific heat?

(c). By how much is the ground-state (T=0) energy per unit volume of the superconductor lower than that of the normal metal, when $H = 0$?