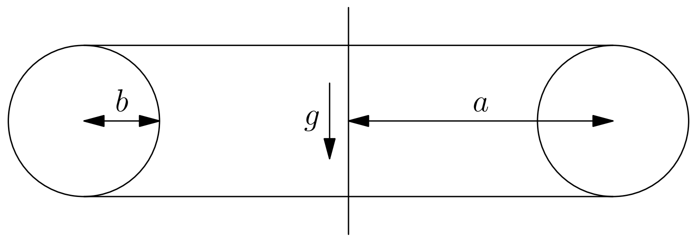
J01M.1—Particle on a Torus

Problem

Find the frequency of small oscillations about uniform circular motion of a point mass that is constrained to move on the surface of a torus (donut) of major radius a and minor radius b whose axis is vertical.



J01M.2—Free Precession of a Planet - 1

Problem

The following two problems relate to a calculation of the angular frequency Ω of free precession of a planet or star whose angular frequency of rotation about its axis is ω. The problems themselves are independent.

Suppose that the density ρ of the object is uniform, and that its shape can be determined by the condition of hydrostatic equilibrium. Deduce an expression for the (small) quantity $\epsilon(\omega, M, r_P)$ that relates the equatorial radius r_E to the polar radius r_P by the form $r_E = r_P(1 + \epsilon)$, where $M \approx 4\pi\rho r_P^3/3$ is the mass of the object.

J01M.3—Free Precession of a Planet - 2

Problem

Suppose the object can be treated as a rigid body whose principal moments of inertia obey $(I_P - I_E)/I_P = \epsilon$ to deduce the angular frequency Ω of free precession in terms of the angular frequency ω of rotation.

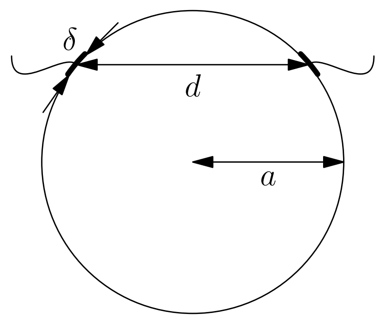
The reader will note that the models of the two problems are somewhat contradictory. However, they work fairly well for the Earth, whose observed free precession period of 430 days (Chandler, 1891) is about 1.6 times that as estimated above. The Chandler wobble is thought to be driven by surface wind and water; see Science **289**, 710 (4 Aug. 2000). First evidence for free precession of a pulsar, PSR B1828-11, has recently been reported by Princeton Ph.D. I.H. Stairs, Nature **406**, 484 (2000), with a period about 1/150 that of the above model. This discrepancy is ascribed to little understood aspects of the superfluid interior of the pulsar.

J01E.1—Resistance Between Two Points on a Disk

Problem

Calculate the resistance between two contacts on the rim of a disk of radius a, thickness $t \ll a$, and conductivity σ, when each (perfectly conducting) contact extends for a small distance δ around the circumference, and the distance along the chord between the contacts is $d \gg \delta$.

The contacts set up semicircular regions of radius $\delta/2$ of nearly uniform potential that extend into the resistive disk.



J01E.2—Betatron

Problem

A betatron is a device in which ultrarelativistic electrons are held in a circle of fixed radius R (taken to be centered on the origin in the x-y plane) by a magnetic field $B_z(r,t)$ while their energy is increased via a changing magnetic flux $d\Phi/dt = \pi R^2\, dB_{z,ave}/dt$ through the circle. Motion of the electrons perpendicular to the circle is prevented by means that need not be considered here.

Deduce the relation between the magnetic field B_z at radius R and the magnetic field $B_{z,ave}$ averaged over the area of the circle. Also deduce the maximum energy $\mathcal{E}$ to which an electron could be accelerated by a betatron in terms of B_z, $dB_{z,ave}/dt$ and R.

Hints: The electrons in this problem are ultrarelativistic, so it is useful to introduce the factor $\gamma = \mathcal{E}/mc^2 \gg 1$, where c is the speed of light. Recall that Newton's second law has the same form for nonrelativistic and relativistic electrons except that in the latter case the effective mass is γm. Recall also that for circular motion the rest frame acceleration is γ^2 times that in the lab frame.

J01E.3—Electromagnetic Wave on a Slab of Dielectric

Problem

A plane electromagnetic pulse $E(z,t) = f(z/c - t)$ is incident from vacuum at $z < 0$ on a dielectric medium that extends from $z = 0$ to $z = a$. The region $z > a$ is also vacuum. The pulse has Fourier components only at frequencies near its central angular frequency ω_0. The index of refraction $n(\omega)$ of the medium is near unity, so reflections at the boundaries can be ignored, and the approximation

$$\omega n(\omega) \approx \omega_0 + \left.\frac{d(\omega n(\omega))}{d\omega}\right|_{\omega_0} (\omega - \omega_0)$$

holds over the relevant frequency bandwidth of the waveform.

Compute the waveform $E(z,t)$ in the dielectric region $0 < z < a$, and in the vacuum region $a < z$.

If ω_0 is chosen to lie between two spectral lines of the medium, which is pumped by lasers at those frequencies into inverted populations, then the group velocity $v_g(\omega_0)$ can be negative, as recently demonstrated by Wang *et al.*, Nature **406**, 277 (2000). Comment on any unusual features of the pulse propagation in this case.

Hint: first discuss the propagation of a monochromatic wave; then consider its impli- cations for the Fourier analysis of the pulse.

J01Q.1—Excitation of a Harmonic Oscillator

Problem

A particle with mass M_1 is moving along the x axis subject to the one-dimensional harmonic-oscillator potential $V(x_1) = \frac{1}{2}M_1\omega^2 x_1^2$. A second particle with mass M_2 is also moving along the x axis. It is free (and in particular does not feel the potential $V(x_1)$) but has an interaction $\lambda\delta(x_1 - x_2)$ with the first particle, where $\delta(x)$ is the Dirac delta function. The total potential is

$$V(x_1, x_2) = \frac{1}{2}M_1\omega^2 x_1^2 + \lambda\delta(x_1 - x_2).$$

Suppose particle 1 is in the ground state of the harmonic oscillator and particle 2 is incoming from $x_2 = -\infty$ with momentum $p > 0$. To first order in $|\lambda|^2$, calculate the probability that as a result of the scattering, particle 1 will be in the first excited state.

J01Q.2—Scattering From Two Delta-function Potentials

Problem

Consider a beam of particles moving in one dimension in a potential $V(x)$ consisting of two delta-functions with strength α separated by a distance a,

$$V(x) = \alpha \left(\delta(x) + \delta(x-a) \right).$$

The particles come in from $x = -\infty$ with momentum $p > 0$. Find the transmission and reflection coefficients. For a given strength $\alpha \neq 0$, find the condition on distance a so that the particles are completely transmitted. Show graphically that the condition has at least one solution.

J01Q.3—Spin in a Magnetic Field

Problem

Consider a particle with spin $S = 1/2$ at rest in a constant magnetic field $\vec{B}$ that is directed along the positive x axis. The Hamiltonian is given by

$$H = g\mu_B \vec{B} \cdot \vec{S}.$$

The spin can be measured in an apparatus that determines S_z to be up or down. At time $t = 0$ the spin is up ($S_z = \hbar/2$). Let T be the time at which the probability of finding the spin to be up is zero for the first time, assuming that no other measurements have been done between $t = 0$ and $t = T$. Suppose instead that the spin is measured repeatedly at constant time intervals T/N with N an integer.

a) What is the probability that the spin is found to be up at all measurements from $t = 0$ up to and including the measurement at $t = T$? Show that this probability approaches one for large N. Estimate its deviation from one.

b) Find the probability for the spin to be up at time $t = T$ while taking into account the possibility that the spin has changed from up to down and back several times at the intermediate measurements between $t = 0$ and $t = T$. Try to simplify the resulting combinatorics.

J01T.1—Containers of Ideal Gas

Problem

Consider two containers filled with the same ideal gas, each initially with the same volume V_1, temperature T_1 and pressure P_1. The volume of one of the containers is subsequently reduced from V_1 to V_2 while keeping the temperature fixed.

a) What is the heat Q yielded by the container during this isothermal compression process?

b) Determine the maximal work W that could be done by subsequently connecting the two containers, such that $V_1 + V_2$ remains constant and no additional heat flows in to or out of the system.

c) Show that W is smaller than Q. Hint: Compare their derivatives with respect to V_2.

J01T.2—Solution of Biomolecules

Problem

Imagine a solution of three types of biomolecules. Type A and type C molecules can form a bound system with energy $-\epsilon_{AC}$ ($\epsilon_{AC} > 0$) relative to $\epsilon = 0$ when they are unbound. Similarly, type B and type C molecules bind with energy $-\epsilon_{BC}$. Only one A or B molecule can bind to a C molecule at a time. Further, ϵ_{AC} and ϵ_{BC} are substantially larger than the energies with which other molecules might be bound at the same place on the type C molecule. The solution is an infinite reservoir of A and B molecules as far as the C molecules are concerned.

a) Determine the grand partition function for this system. Also determine the fractions f_A and f_B of C molecules which have bound an A or a B molecule. You may introduce the chemical potentials, μ_A and μ_B, of A and B molecules.

b) The concentration n_A of A molecules is sufficiently high that in the absence of B molecules, essentially every C molecule binds an A molecule. Obtain an expression for f_A that depends only on the concentrations n_A, n_B, the energies ϵ_{AC} and ϵ_{BC}, and the temperature T the solution.

c) As already remarked, in the absence of B molecules, f_A is close to 1. However, when the concentration n_B of B molecules in solution reaches 1% that of the A molecules in solution, it is observed that f_A drops to 0.1. What is the numerical value of $(\epsilon_{BC} - \epsilon_{AC})/kT$? Make a rough estimate of $\epsilon_{BC} - \epsilon_{AC}$ in electron volts if $T = 300\text{K}$.

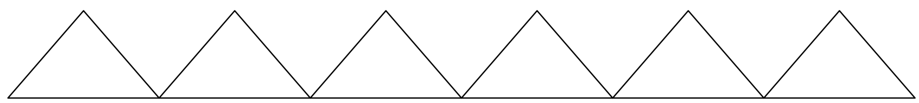
J01T.3—Triangular Chain Ising Model

Problem

Mean field treatments of ferromagnetic systems are quite useful in general, but for certain "frustrated" antiferromagnets this may not be the case. Consider the antiferromagnetic Ising model on the triangular chain shown below, with Hamiltonian

$$H = J \sum_{\langle ij \rangle} s_i s_j,$$

where $J > 0$, the spins s_i have values ± 1, and the sum runs over all nearest neighbor pairs on the lattice (vertices joined by links in the figure).



a) Consider a system of N triangles at temperature T. Assuming that all the top row spins have zero thermal expectation values, write down a mean field theory for the spins. What is the transition temperature?

b) Consider a system consisting of a single triangle at $T = 0$. How many ground states (minimum energy configurations) does it have?

c) Consider a system of N triangles at $T = 0$. How many ground states does the chain have, supposing free boundary conditions?

d) Calculate the correlation $\langle s_i s_j \rangle$, averaged over the ground states for two spins on the bottom row (N triangles). Is this consistent with the answer in part a)?

(Hint: In parts c) and d) fairly simple counting arguments can give exact results.)