Anisotropy in the Integer Quantum Hall Effect

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Bendeguz Offertaler

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Supervisor: Dr. Barry Bradlyn
Second reader: Professor Shivaji Sondhi

This paper represents my own work in accordance with University regulations.
/s/ Bendeguz Offertaler

Abstract

The purpose of this paper is to better understand the role of anisotropy in the integer quantum Hall effect. We analyze two systems discussed in the literature. The first is that of an electron moving in a plane with a perpendicular magnetic field and an anisotropic mass tensor. The second is that of an electron moving in three dimensions with a confining harmonic potential in the z-direction and a magnetic field tilted relative to the z-axis. Our primary focus is on the tilted field system in the regime of weak magnetic field tilt and strong confining potential. In the latter limit, the 3D system approaches an effective 2D system. The main tool we use in our analysis is linear response theory, which examines how observables change in response to small perturbations in external fields. We analyze both the Hall conductivity, which quantifies how the current changes in response to an applied electric field, and the Hall viscosity, which quantifies how the stress tensor changes in response to a strain. These observables are richly characterized in the isotropic case but are less well understood in systems without rotational symmetry in the plane. Our results for the Hall viscosity and projected stress tensor operators indicate that the 2D limit of the tilted field system cannot be fully characterized by an effective mass anisotropy. There are multiple physical effects that have an intrinsically 3D origin and do not have analogs in the simplest bona fide 2D system. Our work suggests that the projection of the tilted field system into two dimensions is more interesting than previously believed.

1 Introduction

Many of the exciting discoveries made in condensed matter physics over the last forty years fall into the category of quantum Hall effects (QHE). The system that is the focus of the QHE is simple: a strong perpendicular magnetic field is applied to a planar conducting surface. It is equivalent to a two-dimensional electron gas (2DEG) in an external magnetic field. The resulting phases of matter have exotic properties and remarkable variety.

The first major discovery came in 1980 when K. von Klitzing, G. Dorda and M. Pepper [9] measured the Hall resistance of samples cooled below 2 K and subject to $\sim$ 15 T magnetic fields. They discovered that the Hall resistance, which relates the current to the perpendicular voltage, exhibits plateaus at $1/\nu$ multiples of the flux quantum $\phi_0 = \frac{2\pi\hbar}{e^2}$ where ν takes integral values with an accuracy of one part in one billion.[1] Furthermore, the longitudinal resistance effectively vanishes

[1] The high experimental accuracy relies in part on the fact that in two dimensions the Hall resistance (an extensive quantity) and the Hall resistivity (an intensive quantity) are the same. Namely, $V_x \equiv (LE_x) = \rho_{xy}(LJ_y) \equiv \rho_{xy} I_y$, where L is the length along the perpendicular direction. Therefore, the Hall resistance does not depend on the macroscopic parameters of the sample, which would be very difficult to measure with errors of order 10^{-9}.

on the plateaus. Both results were unexpected. It was surprising to find precise quantization of a macroscopic property in a system with impurities and the vanishing longitudinal resistance indicated that the sample behaves as a "superconductor" on the plateaus, but not in the transition between plateaus.

In the decade that followed, new experimental discoveries made remarkable additions to the picture first presented by von Klitzing. D. C. Tsui, H. L. Störmer and A. C. Gossard found that at sufficiently high magnetic fields, new plateaus emerge where the resistance is a fractional multiple of the flux quantum [19]. The fractional quantum Hall Effect (FQHE) has many unusual properties not found in the integer quantum Hall effect (IQHE), like fractional charge fractional statistics, and a complex hierarchy of possible states [4].

Meanwhile, the theoretical condensed matter community developed a framework to understand the different properties and states that were observed. The robustness of the multitude of states and their observable properties can be attributed to the existence of gaps in the energy spectrum, generated by the external magnetic field in the case of the IQHE and by the electron interactions in the FQHE. The existence of a protective gap is often used to define a topological phase (see, e.g., Section 1.1 of [3]), so the quantum Hall states can be thought of as distinct topological phases with transitions induced by a change in the filling fraction of electrons or other parameters of the system. Early theoretical work on the QHE mainly used microscopic wavefunction based arguments (see for instance the work of R. Laughlin, [10], [11]), but later work using an effective field theory approach reproduced many of the same conclusions and added further insight as well. For good, accessible notes on the quantum Hall effect, see [4] and [18].

Continuing a tradition established by the von Klitzing experiment, a common approach to understanding quantum Hall states is to look at the behavior of linear response functions. Two response functions discussed extensively are the conductivity,[2] which captures how the current responds to an electric field, and the viscosity, which captures how the stress-energy tensor responds to a time-varying strain. Usually, it is the *Hall* conductivity and *Hall* viscosity– the antisymmetric components of the respective response functions– which are particularly interesting.

The behavior of linear response functions in rotationally symmetric quantum Hall systems is much better understood than in systems with broken rotational symmetry. For instance, although the Hall conductivity is known to be topologically invariant in systems with or without rotational symmetry [12], there is an elegant argument for the quantization of the Hall viscosity[3] that relies on rotational symmetry [15]. It is not known whether a similar result holds for the Hall viscosity in anisotropic systems.

In this paper, we examine two simple quantum Hall systems with anisotropy. We calculate the expectation values and projected forms of different observables, like the Hall conductivity, Hall viscosity and stress tensor, and examine their properties. Although we do not directly look at topological or geometric properties of these quantum Hall states, the hope is that the results presented here may prove useful for future work along such lines.

For simplicity, we ignore electron-electron interactions, which means we are working exclusively with the IQHE. Furthermore, we ignore disorder, which means we do not deal directly with the

[2]It is more common in quantum Hall literature to talk about the Hall conductivity than the Hall resistivity, which is what early experiments investigated. Since the resistivity and conductivity are inversely related, they contain the same information.

[3]As in the case of the Hall conductivity, for isotropic systems the Hall viscosity can be written as an integral of a Berry curvature. Theorems from differential geometry state that under certain conditions the integral of curvature over a surface is an integer.

formation of plateaus in different observables.[4] The first system takes the isotropic Hamiltonian in the plane, which is given by

$$H_I = \frac{1}{2m} \sum_{i=1}^{N} \sum_{a=1}^{2} \pi_a^i \pi_a^i, \qquad \nabla \times \vec{A} = B\hat{z} \tag{1}$$

where $\vec{\pi}^i = \vec{p}^i + e\vec{A}(\vec{x}^i)$, and couples it to an anisotropic mass tensor. Thus, the first Hamiltonian of interest, which we call the band mass Hamiltonian, is

$$H_{BM} = \frac{1}{2} \sum_{i=1}^{N} \sum_{a,b=1}^{2} \tilde{m}_{ab} \pi_a^i \pi_b^i, \qquad \nabla \times \vec{A} = B\hat{z} \tag{2}$$

where $\tilde{m}_{ab}$ is a symmetric invertible matrix. The band mass Hamiltonian introduces anisotropy in a particularly simple way and has experimental plausibility. For instance, it is a well-known result that an electron moving in a periodic potential behaves like a free-particle with an adjusted effective mass [1]. For a crystal with different spacing in the x and y directions, the effective mass becomes an anisotropic effective mass tensor.

The second system, which is central to our analysis, places an electron in a confining potential $V(z) = \frac{1}{2}m\omega_0^2 z^2$ and a magnetic field $\vec{B} = B_x\hat{x} + B_z\hat{z}$. The Hamiltonian, which we call the tilted field Hamiltonian, is then

$$H_{TF} = \sum_{i=1}^{N} \left[\frac{1}{2m} \sum_{\mu=1,2,3} \pi_\mu^i \pi_\mu^i + \frac{1}{2} m\omega_0^2 (z^i)^2 \right], \qquad \nabla \times \vec{A} = B_x\hat{x} + B_z\hat{z} \tag{3}$$

We are interested in the limit $\omega_0 \to \infty$ and $B_x \to 0$. In this limit, the anisotropy is weak, and because the confining potential is strong, the 3D system becomes an effective 2D system similar to the typical quantum Hall system. Both systems we analyze have been discussed previously in the literature (see, e.g., [13], [20], [21]), and some of our work overlaps with previous findings.

A quick note: the Hamiltonians given in equations (1) $-$ (3) introduce an index convention we will adhere to throughout the paper. Upper Roman indices starting from i label the different particles in the system. Lower Roman indices starting from a label the spatial coordinates 1 and 2, shorthand for x and y. Finally, lower Greek indices label the spatial coordinates 1, 2 and 3, shorthand for x, y and z. If it's ambiguous whether we are working in 2 or 3 dimensions, we use Greek indices. From now on, we use Einstein summation convention for the spatial indices but continue to include the sums over particles explicitly.

The remainder of the paper is structured as follows. Section 2 reviews basic features of single particle quantum mechanics in an electromagnetic field. Section 3 presents an overview of linear response theory. Section 3.1 derives the central result relating perturbations in observables to response functions, and the expression for the response functions in terms of the commutator of an observable and a perturbing operator. Section 3.2 discusses important properties of linear response functions. Section 4 defines the Hall conductivity and Hall viscosity as response functions corresponding to perturbations in the electric field and a uniform strain, respectively. Sections 5 and 6 analyze the two anisotropic systems. The first subsections present derivations of the energy eigenstates. The second, third and fourth subsections compute the Hall conductivity, stress-energy

[4]There is a quaint argument using Galilean boosts and the transformation properties of the electromagnetic field that shows the Hall conductivity of systems without disorder is a linear function of the magnetic field. See Section 1.1.3. of [4]. Thus, the inclusion of disorder is necessary for plateau formation.

tensor and Hall viscosity. Section 7 features a unifying discussion in which effective anisotropic mass tensors are extracted for the tilted field using the different observables. We discuss the limitations of such an approach. Finally, the appendices contain a derivation of a Fourier transform property and extra results that we calculated but do not discuss in the paper.

2 Single particle quantum mechanics in an electromagnetic field

Since we are interested in the behavior of non-relativistic and non-interacting electrons in Hall systems, we begin with a review of single particle quantum mechanics in an electromagnetic field. Recall that the Lagrangian for a particle with charge q, moving in an electromagnetic field specified by a scalar potential $\phi(x,t)$ and vector potential $\vec{A}(x,t)$, is given by

$$L = \frac{1}{2}m\dot{\vec{x}}^2 + q\dot{\vec{x}} \cdot \vec{A} - q\phi \tag{4}$$

The conjugate momentum is $\vec{p} = \frac{\partial L}{\partial \dot{\vec{x}}} = m\dot{\vec{x}} + q\vec{A}$ and so the Hamiltonian becomes

$$H = \vec{p} \cdot \dot{\vec{x}} - L = \frac{1}{2m}\dot{\vec{x}}^2 + q\phi = \frac{1}{2m}\left(\vec{p} - q\vec{A}\right)^2 + q\phi \tag{5}$$

Notice that the vector potential does not appear explicitly in the Hamiltonian when it is written in terms of the velocity $\dot{\vec{x}}$. This reflects the fact that magnetic fields do no work.

When we switch to quantum mechanics, we promote x_μ and p_ν to operators satisfying the canonical commutation relation $[x_\mu, p_\nu] = i\delta_{\mu\nu}$. Because the vector potential $\vec{A}$ is defined only up to a gauge transformation $\vec{A} \to \vec{A} + \nabla\chi$, it is not physical in the same sense that $\vec{x}$ is. The same holds for gauge dependent functions of $\vec{A}$, including $\vec{p}$, which transforms like $\vec{p} \to \vec{p} + q\nabla\chi$.[5] It is thus convenient to define a gauge invariant momentum, $\vec{\pi} = \vec{p} - q\vec{A}$. It obeys $[x_\mu, \pi_\nu] = i\delta_{\mu\nu}$ and

$$[\pi_\mu, \pi_\nu] = [p_\mu - qA_\mu, p_\nu - qA_\nu] = -q\left([p_\mu, A_\nu] + [A_\mu, p_\nu]\right) = iq\left(\partial_\mu A_\nu - \partial_\nu A_\mu\right) \tag{6}$$

Using $B_\mu = \epsilon_{\mu\nu\rho}\partial_\nu A_\rho$ and $\epsilon_{\mu\nu\rho}\epsilon_{\mu\sigma\tau} = \delta_{\nu\sigma}\delta_{\rho\tau} - \delta_{\nu\tau}\delta_{\rho\sigma}$, we therefore find $[\pi_\mu, \pi_\nu] = iq\epsilon_{\mu\nu\rho}B_\rho$.

Looking at dynamics in the Heisenberg picture, we find

$$\frac{\partial x_\mu}{\partial t} = i[H, x_\mu] = \frac{i}{2m}[\pi_\nu\pi_\nu, x_\mu] = \frac{\pi_\mu}{m} \tag{7}$$

and

$$\frac{\partial \pi_\mu}{\partial t} = i[H, \pi_\mu] = \frac{i}{2m}[\pi_\nu\pi_\nu, \pi_\mu] + iq[\phi, \pi_\mu] = q\left[\epsilon_{\mu\nu\rho}\frac{\pi_\nu}{m}B_\rho - \partial_\mu\phi\right] \tag{8}$$

Note that the right hand side of equation (8) is the Lorentz force, $\vec{F} = q(\vec{v} \times \vec{B} + \vec{E})$. In other words, $\vec{\pi}$ behaves like the physical momentum classically should. We therefore identify $\vec{\pi}$ as the physical momentum operator. For instance, when we deduce the stress tensor operator in later sections from a continuity equation for the momentum density, we write the momentum density operator for N particles, which would be $\vec{g}(\vec{r},t) = \frac{1}{2}\sum_{i=1}^{N}\{\vec{p}^i, \delta(\vec{r} - \vec{x}^i)\}$ without a magnetic field, as $\vec{g}(\vec{r},t) = \frac{1}{2}\sum_{i=1}^{N}\{\vec{\pi}^i, \delta(\vec{r} - \vec{x}^i)\}$. The anti-commutator brackets ensure $\vec{g}(\vec{r},t)$ is Hermitian.

[5] Also worth mentioning is the behavior of the wavefunction under a gauge transformation. Since $\langle\vec{x}|\vec{p} + q\nabla\chi|\psi\rangle = \left(\frac{\hbar}{i}\frac{\partial}{\partial\vec{x}} + q\nabla\chi\right)\langle\vec{x}|\psi\rangle$, we see that the wavefunction changes via $\psi(\vec{x}) \to e^{\frac{iq\chi}{\hbar}}\psi(\vec{x})$ under a gauge transformation. We may use this property to get rid of undesirable factors of the form $e^{if(\vec{x})}$ in our wavefunctions.

When the charged particle is confined to a plane (the usual setting for the Hall effect), the only relevant component of the magnetic field is the one perpendicular to the surface. Classically, this is because the parallel components of the magnetic field produce forces perpendicular to the surface, which are canceled by the constraining forces. Quantum mechanically, confining the particle to the plane means working exclusively with coordinates x, y, canonical momenta p_x, p_y and physical momenta π_x, π_y. The only commutator that depends on the magnetic field is $[\pi_a, \pi_b] = iq\epsilon_{abz}B_z$, and it does not depend on the parallel components of the magnetic field. Introducing the completely antisymmetric 2D Levi-Civita tensor ϵ_{ab} satisfying $\epsilon_{12} = 1$, and writing $B \equiv B_z$, the commutator becomes $[\pi_a, \pi_b] = iqB\epsilon_{ab}$.

Returning briefly to the topic of gauge transformations, we can use the gauge freedom to simplify calculations. In the case of a 2D system, two gauge choices are common: the Landau gauge, for which $\vec{A} = B_z x\hat{y}$, and the symmetric gauge, for which $\vec{A} = B_z(-y/2\hat{x} + x/2\hat{y})$. When tackling the tilted field Hamiltonian, we use a generalization of the Landau gauge to solve for the energy eigenfunctions in Section 6.1.1, and a generalization of the symmetric gauge to solve for the "lowest angular momentum" ground state eigenfunction, which we use to deduce an effective anisotropy tensor in Section 7.1. Despite its usefulness, picking a convenient gauge sacrifices explicit gauge invariance of our results. Thus, it is sometimes better to use gauge invariant operator methods. We do both in this paper.

Finally, since we are interested in the behavior of an electron (gas), moving forward we set $q = -e$. Following B. Bradlyn in [3], we also work in units where $\hbar = e = 1$, in which case the flux quantum simplifies to $\phi_0 = 2\pi$.

3 Linear response

In linear response theory, we probe a stationary system by turning on a weak external field and seeing how the observables evolve. The formalism can be derived using time dependent perturbation theory in the interaction picture. The approach taken in this section closely follows [3] and draws on [17].

3.1 General theory

Consider a system with unperturbed Hamiltonian H_0 and unperturbed stationary density matrix $\rho(t) = \rho_0$. Recall that the density matrix can be defined as a statistical sum of pure states, $\rho(t) = \sum_i a_i |\psi_i(t)\rangle \langle\psi_i(t)|$, from which two properties follow directly. First, the expected value of an observable $\mathcal{O}$ is given by $\langle\mathcal{O}(t)\rangle = \mathrm{Tr}(A\rho(t))$. Second, ρ evolves according to the equation $i\frac{\partial\rho}{\partial t} = [H_0, \rho(t)]$. Equivalently, we may write $\rho(t) = U_0(t, t_0)\rho(t_0)U_0^\dagger(t, t_0)$ where the unitary evolution operator $U(t, t_0)$ satisfies $U(t_0, t_0) = 1$ and the Schrödinger equation:

$$i\frac{\partial U(t, t_0)}{\partial t} = H_0 U(t, t_0) \tag{9}$$

Thus, for our unperturbed density matrix to be stationary we require $[\rho_0, H_0] = 0$.

We turn on perturbing fields so that the Hamiltonian becomes $H(t) = H_0 + H_1(t)$ with $H_1(t) = f_n(t)B_n e^{\epsilon t}$ (sum over n). The functions $f_n(t)$ capture the time-dependence of the nth field, whose coupling to the Hamiltonian is represented by the operators B_n. The exponential $e^{\epsilon t}$ ensures that the perturbations turn off as $t \to -\infty$. We can alternatively think of the exponential as a tool for making our integrals convergent. We take the limit $\epsilon \to 0^+$ at the end.

The key question is how the expectation values for a set of observables A_n change in response to the external fields. Thus, we need $\delta\langle A_m(t)\rangle = \mathrm{Tr}(A_m(\rho(t) - \rho_0))$, where $\rho(t)$ now evolves according to $i\frac{\partial \rho}{\partial t} = [H, \rho]$. We use time-dependent perturbation theory to approximate $\rho(t)$ to linear order in the strength of the perturbations. It is natural to work in the interaction picture, where the operators are made time-dependent according to

$$\mathcal{O}_I = e^{iH_0(t-t_0)}\mathcal{O}_S e^{-iH_0(t-t_0)} \tag{10}$$

From $\langle \mathcal{O}(t)\rangle = \mathrm{Tr}(\mathcal{O}_S\rho_S(t)) = \mathrm{Tr}(\mathcal{O}_I\rho_I(t)) = \mathrm{Tr}(e^{iH_0(t-t_0)}\mathcal{O}_S e^{-iH_0(t-t_0)}\rho_I)$, we see that $\rho_I = e^{iH_0(t-t_0)}\rho_S e^{-iH_0(t-t_0)}$. We may therefore write $\rho_I = U_I(t,t_0)\rho(t_0)U_I^\dagger(t,t_0)$ where we introduce the interaction evolution operator, $U_I(t,t_0) = e^{iH_0(t-t_0)}U(t,t_0)$. It obeys its own Schrödinger equation:

$$\begin{aligned} i\frac{\partial U_I(t,t_0)}{\partial t} &= -e^{iH_0(t-t_0)}H_0U(t,t_0) + e^{iH_0(t-t_0)}HU(t,t_0) \\ &= e^{iH_0(t-t_0)}(H-H_0)e^{-iH_0(t-t_0)}e^{iH_0(t-t_0)}U(t,t_0) = H_1(t)U_I(t,t_0) \end{aligned} \tag{11}$$

where $H_1(t)$ is now evaluated in the interaction picture. We may formally solve for $U_I(t,t_0)$ using a Dyson series. We have:

$$U_I(t,t_0) = 1 - i\int_{t_0}^{t} dt_1 H_1(t_1) + (-i)^2\int_{t_0}^{t}\int_{t_0}^{t_1} dt_1dt_2H_1(t_1)H_1(t_2) + \dots \tag{12}$$

To linear order, the change in the expectation value of A_n becomes

$$\begin{aligned} \delta\langle A_m\rangle &= \mathrm{Tr}\left(A_m(t)\left(\left[1 - i\int_{t_0}^{t} dt' H_1(t')\right]\rho(t_0)\left[1 + i\int_{t_0}^{t} dt' H_1(t')\right] - \rho(t_0)\right)\right) \\ &= -i\int_{t_0}^{t} dt' \mathrm{Tr}\left([A_m(t), H_1(t')]\rho(t_0)\right) \end{aligned} \tag{13}$$

Remembering that we want to take $t_0 \to -\infty$ so that $\rho(t_0) \to \rho_0$, and $H_1(t) = f_n(t)B_n(t)e^{\epsilon t}$, we finally get

$$\delta\langle A_m(t)\rangle = \int_{-\infty}^{\infty} dt' \chi_{mn}(t-t')f_n(t') \tag{14}$$

where the response function $\chi_{mn}(t)$ is given by

$$\chi_{mn}(t) = -i\theta(t)\lim_{\epsilon\to 0^+}\langle [A_m(t), B_n(0)]\rangle_0 e^{-\epsilon t} \tag{15}$$

Equations (14) and (15) together form the key result of this section. Note that by introducing $\theta(t)$, the Heaviside function, we are able to write ∞ for the upper-bound of the integral instead of t. Since the Heaviside function ensures that the observables only react to the perturbation at times $t' < t$, we can interpret it as imposing causality. Also, we were able to write $\langle [A_m(t), B_n(t')]\rangle_0 = \langle [A_m(t-t'), B_n(0)]\rangle_0$ because the unperturbed Hamiltonian H_0 and unperturbed density matrix ρ_0 are time-translation invariant (i.e., neither depends on time). Incidentally, time-translation invariance implies that energy is conserved in the unperturbed system. The formalism we have developed must be modified if it is to be applied to nonconservative systems.

In the event that the observables A_m are position dependent (e.g., they correspond to current or momentum densities) and the perturbing Hamiltonian is of the form $H_1 = \int d^dx B_n(x')f_n(x',t')e^{\epsilon t}$, the above results are straightforward to generalize. We have:

$$\delta\langle A_m\rangle(\vec{x},t) = \int_{-\infty}^{\infty} dt' \int d^dx' \chi_{mn}(\vec{x},\vec{x}',t-t')f_n(\vec{x}',t') \tag{16}$$

where d is the dimension of the space and the response function is given by

$$\chi_{mn}(\vec{x},\vec{x}',t) = -i\theta(t)\langle [A_m(\vec{x},t), B_n(\vec{x}',0)]\rangle_0 e^{-\epsilon t} \tag{17}$$

3.2 Properties of linear response functions

Because $\delta\langle A_m(t)\rangle$ is given as a convolution in time of the response function and the perturbing function, it is useful to express results in the frequency domain. Taking a Fourier transform of both sides of (14), we have

$$\delta\langle A_m(\omega)\rangle = \chi_{mn}(\omega) f_n(\omega) \tag{18}$$

$$\chi_{mn}(\omega) = -i \lim_{\epsilon\to 0^+}\int_0^\infty dt e^{i\omega^+ t}\langle [A_m(t), B_n(0)]\rangle_0 \tag{19}$$

and $\omega^+ \equiv \omega + i\epsilon$. From now on, the limit $\epsilon \to 0^+$ will be implicit.

In the event that our operators are position dependent– so that we use equations (16) and (17) to characterize the linear response– and the system is translationally invariant, we can also take a spatial Fourier transform of the expectation value of the perturbed operator. We find (see Appendix A for the details)

$$\delta\langle A_m\rangle(\vec{q},\omega) = \frac{1}{V}\chi_{mn}(\vec{q},-\vec{q},\omega) f_n(\vec{q},\omega) \tag{20}$$

$$\chi_{mn}(\vec{q},\vec{q}',\omega) = -i\int_0^\infty dt e^{i\omega^+ t}\langle [A_m(\vec{q},t), B_n(\vec{q}'), 0)]\rangle \tag{21}$$

where V is the volume (or area) of the space. In such cases, we may choose to redefine $\chi_{mn}(\vec{q},\omega) \equiv \chi_{mn}(\vec{q},-\vec{q},\omega)$, as, for instance, we do with the Hall conductivity in the next section.

Using the time Fourier component of the response function, we deduce two important properties. Firstly, at zero frequency, only the symmetric part of the response function is dissipative. Secondly, for certain systems that are time reversal invariant, the antisymmetric component of the response function is zero. To derive both properties, we assume that $B_m = A_m$, an assumption that will be satisfied by our formulas for the Hall conductivity and Hall viscosity.

The time integral of the expectation value of the perturbing Hamiltonian is a measure of the dissipated energy W, and is given by

$$\begin{aligned} W &= \int_{-\infty}^{\infty} dt\,\langle H_1(t)\rangle = \int_{-\infty}^{\infty} dt f_m(t)\langle A_m(t)\rangle \\ &= \int_{-\infty}^{\infty} dt\left(\frac{1}{2\pi}\int_{-\infty}^{\infty} d\omega e^{-i\omega t} f_m(\omega)\right)\left(\frac{1}{2\pi}\int_{-\infty}^{\infty} d\omega' e^{-i\omega' t}\langle A_m(\omega')\rangle\right) \\ &= \frac{1}{2\pi}\int_{-\infty}^{\infty} d\omega f_m(-\omega)\langle A_m(\omega)\rangle \end{aligned} \tag{22}$$

where in the second line we wrote $f_m(t)$ and $\langle A_m(t)\rangle$ in terms of their Fourier expansions, then evaluated the t integral to get a delta function, which let us evaluate the ω' integral. Using equation (18), we can therefore write

$$W = \frac{1}{2\pi}\int_{-\infty}^{\infty} d\omega f_m(-\omega)\chi_{mn}(\omega) f_n(\omega) \tag{23}$$

It is clear that $f_m(0)\chi_{mn}(0)f_n(0) = 0$ if $\chi_{mn}(\omega)$ is antisymmetric under $n \leftrightarrow m$. Thus, the antisymmetric component of the response function does not contribute to dissipation for low frequency perturbations. In particular, neither the Hall conductivity nor the Hall viscosity are dissipative.

To deduce the second property, we look at the the behavior of the response function under time reversal. We define the time reversed response function $\chi^T_{mn}(\omega)$ to be $\chi_{mn}(\omega)$ evaluated in the time reversed ground states. Recall that the time reversal operator T is anti-linear and anti-unitary. Anti-linearity means that T acting on a complex number takes its complex conjugate. Another useful property is that, given states $|\alpha\rangle$, $|\beta\rangle$ and operator $\mathcal{O}$, we have $\langle\beta|\mathcal{O}|\alpha\rangle = \langle T\alpha|T\mathcal{O}^\dagger T^{-1}|T\beta\rangle$ (see, e.g., [16]). For $\mathcal{O} = I$, we get $\langle\beta|\alpha\rangle = \langle T\alpha|T\beta\rangle$, which is just the anti-unitary property.

Let us assume that A_m has a definite signature under time reversal; namely, $T^{-1}A_mT = (-1)^{\alpha_m}A_m$. For instance, x and p have positive ($\alpha = 0$) and negative ($\alpha = 1$) signatures, respectively. In the event that the unperturbed Hamiltonian is time reversal invariant (i.e., $THT^{-1} = H$, which implies $TA(t)T^{-1} = T(e^{iHt}A(0)e^{-iHt})T^{-1} = (-1)^\alpha A(-t)$), we can deduce a condition on the response function. We have

$$\begin{aligned}
\chi^T_{mn}(\omega) &\equiv -i\int_0^\infty dte^{i\omega^+t}\,\langle T\psi_0|[A_m(t), A_n(0)]|T\psi_0\rangle \\
&= -i(-1)^{\alpha_n+\alpha_m}\int_0^\infty dte^{i\omega^+t}\,\langle[A_m(-t), A_n(0)]^\dagger\rangle_0 \\
&= -i(-1)^{\alpha_n+\alpha_m}\int_0^\infty dte^{i\omega^+t}\,\langle A_n(t), A_m(0)\rangle_0 \\
&= (-1)^{\alpha_n+\alpha_m}\chi_{nm}(\omega)
\end{aligned} \tag{24}$$

If, additionally, the ground state is time reversal invariant (i.e., $T|\psi_0\rangle = |\psi_0\rangle$), we can see by the first line above that $\chi^T_{mn}(\omega) = \chi_{mn}(\omega)$. Finally, if the operators A_m all have the same signature (so that $(-1)^{\alpha_n+\alpha_m} = 1$), which is true for the response functions we will examine, we arrive at the result $\chi_{mn}(\omega) = \chi_{nm}(\omega)$.

The second property explains why a magnetic field is necessary for the Hall conductivity and Hall viscosity to be non-zero, because the magnetic field breaks time reversal symmetry. To see this, recall that all electrons in a uniform magnetic field orbit in the same orientation; reversing the momentum of an electron does not send it back along its cyclotron orbit. Thus, the quantum Hall system is a perfect set-up for investigating the Hall conductivity and Hall viscosity.

4 Hall conductivity and Hall viscosity in the language of linear response

In this section, we apply the formalism developed in the previous section. We introduce two response functions, the conductivity and the viscosity. By taking the antisymmetric parts of both, we derive expressions for the Hall conductivity and Hall viscosity. We closely follow the approaches taken in [2], [3].

4.1 Conductivity

The conductivity tensor captures how the current density responds to a weak electric field. We work in a gauge where $\phi = 0$, so that the electric field is given by $\vec{E} = -\partial_t\vec{A}$. We effect the perturbation by making a small variation in the vector potential.

We define our observable of interest, the current density, by the variation of the action with respect to the vector potential:

$$j^\mu(\vec{x}, t) \equiv -\frac{\delta}{\delta A_\mu(\vec{x}, t)}\int dt' H_1(t') \tag{25}$$

so that the perturbing Hamiltonian to linear order in A_μ is given by

$$H_1(t) = -\int d^d x j_\mu(\vec{x},t) A_\mu(\vec{x},t) \tag{26}$$

Note, the perturbing Hamiltonian given above has second order corrections because j^μ depends on A_μ. We discuss the consequences of this subtlety below. Proceeding with our formalism in the frequency domain, we have

$$\delta \langle j_\mu \rangle (\vec{x},\omega) = \int d^d x' i\omega \sigma_{\mu\nu}(\vec{x},\vec{x}',\omega) A_\nu(\vec{x}',\omega) \tag{27}$$

where the conductivity is given by

$$\sigma_{\mu\nu}(\vec{x},\vec{x}',\omega) = \frac{1}{\omega^+}\int_0^\infty dt e^{i\omega^+ t} \langle [j_\mu(\vec{x},t), j_\nu(\vec{x}',0)]\rangle_0 \tag{28}$$

The extra factor of $i\omega$ in equation (27) is included because, using the Fourier transform of $\vec{E} = -\partial_t \vec{A}$, we see

$$\delta \langle j_\mu \rangle (\vec{x},\omega) = \int d^d x' \sigma_{\mu\nu}(\vec{x},\vec{x}',\omega) E_\nu(\vec{x}',\omega) \tag{29}$$

which is the standard relation between current density, electric field, and conductivity.[6]

Because the current density operator depends on the gauge, just like the physical momentum operator introduced in Section 2, the current density operator after perturbation is different from the current density operator before perturbation. In particular, the unperturbed expectation value of the perturbed current density, $\langle j_\mu \rangle_0$, is non-zero and is proportional to $A_\mu(\vec{x})$. Thus, our expression for $\langle j_\mu \rangle = \langle j_\mu \rangle_0 + \delta \langle j_\mu \rangle$ includes an extra so-called "contact term" that may be incorporated into the conductivity tensor via a term proportional to $\delta_{\mu\nu}$. If we were interested in the full conductivity, we would have to explicitly determine the contact term. However, since $\delta_{\mu\nu}$ is symmetric and does not contribute to the antisymmetric Hall conductivity, the contact term is a subtlety we can ignore.

Finally, we consider the long-wavelength behavior of the conductivity, which fully characterizes the conductivity in a homogeneous system (like the ones we are considering). Using translational invariance, we can apply the spatial Fourier transform result from the previous section. We find that the conductivity in the momentum domain is

$$\sigma_{\mu\nu}(\vec{q},\omega) = \frac{1}{\omega^+}\int_0^\infty dt e^{i\omega^+ t} \langle [j_\mu(\vec{q},t), j_\nu(-\vec{q},0)]\rangle_0 \tag{30}$$

Long wavelength behavior corresponds to $\sigma_{\mu\nu}(\vec{q}=\vec{0},\omega)$, which is just the integral of the conductivity over the whole space. Likewise for $j_\mu(\vec{q}=\vec{0},t)$. Writing the integrated current density as $\int d^d x j_\mu(\vec{x},t) = J_\mu(t)$, the result for the integrated conductivity (which we may call the conductance) becomes

$$\sigma_{\mu\nu}(\omega) = \frac{1}{\omega^+}\int_0^\infty dt e^{i\omega^+ t} \langle [J_\mu(t), J_\nu(0)]\rangle_0 \tag{31}$$

Since the systems we consider are homogeneous, the conductivity is the conductance divided by the total volume.[7]

[6] If we set $\sigma_{\mu\nu}(x,x') = \tilde{\sigma}_{\mu\nu}(x)\delta(x-x')$, equation (29) yields $\delta\langle j_\mu\rangle(x) = \tilde{\sigma}_{\mu\nu}(x)E_\nu(x)$, which is perhaps a more familiar relation between current density, electric field and conductivity. Equation (27) is more general, however, since it allows for non-local contributions to the current density.

[7] Note, we use $\sigma_{\mu\nu}$ to denote both the conductivity and the conductance. We could choose to write $\Sigma_{\mu\nu}$ for the conductance, in keeping with the lowercase/uppercase for intensive/integrated operators convention. However, writing $\sigma_{\mu\nu}$ for both is conventional. It should be clear from context which one we refer to.

4.2 Viscosity

The second response function of interest, the viscosity tensor, captures how the stress tensor responds to a strain in the system. The strain can be thought of as a generic shearing/dilating coordinate transformation. We consider the case of a spatially uniform strain.

Let us begin with a Hamiltonian describing a collection of interacting electrons[8] moving in an electromagnetic field. It is given by

$$H_0 = \frac{1}{2m}\sum_{i=1}^{N}\pi_\mu^i\pi_\mu^i + \frac{1}{2}\sum_{i\neq j}V(\vec{x}^i - \vec{x}^j) \tag{32}$$

Consider a perturbation to the Hamiltonian in which the coordinates transform as $x_\mu \to \Lambda^T_{\mu\nu}x_\nu$. (Writing the transformation as a transpose is conventional.) In order to maintain the canonical commutation relations, we require that the momenta transform according to $p_\mu \to \Lambda^{-1}_{\mu\nu}p_\nu$ in the case of no magnetic field. The natural generalization is to require $\pi_\mu \to \Lambda^{-1}_{\mu\nu}\pi_\nu$ once a magnetic field is introduced. We require only that $\Lambda_{\mu\nu}$ be invertible.

The Hamiltonian describing the system subject to a time-dependent strain is then given by

$$H_\Lambda(t) = \frac{1}{2m}g^{\mu\nu}(t)\sum_{i=1}^{N}\pi_\mu^i\pi_\nu^i + \frac{1}{2}\sum_{i\neq j}V(\Lambda^T(t)(\vec{x}^i - \vec{x}^j)) \tag{33}$$

where $g^{\mu\nu}(t) = \Lambda^{-1}_{\alpha\mu}(t)\Lambda^{-1}_{\alpha\nu}(t)$ behaves like a metric; we can also define its inverse $g_{\mu\nu}(t) = \Lambda_{\mu\beta}(t)\Lambda_{\nu\beta}(t)$.

We next want to find operators $J_{\mu\nu}$ that generate the strain (i.e., in the same way that the angular momenta generate rotations). Specifically, if we write $\Lambda^{\mu\nu} = e^{\lambda_{\mu\nu}}$, we want $S(t) = e^{-i\lambda_{\mu\nu}J_{\mu\nu}}$ to satisfy $S(t)x^i_\mu S(t)^{-1} = \Lambda^T_{\mu\nu}x^i_\nu$ and $S(t)\pi^i_\mu S(t)^{-1} = \Lambda^{-1}_{\mu\nu}\pi^i_\nu$. It follows then that $S(t)H_0S^{-1}(t) = H_\Lambda(t)$. By expanding the transformation conditions to linear order, we deduce that the strain generators must obey the commutation relations $i[J_{\mu\nu}, \pi^i_\alpha] = \delta_{\alpha\mu}\pi^i_\nu$ and $i[J_{\mu\nu}, x^i_\alpha] = -\delta_{\alpha\nu}x^i_\mu$. Commuting $J_{\rho\sigma}$ with the commutators, using the Jacobi identity to rearrange, and matching the resulting terms, we find that the strain generators also satisfy

$$i[J_{\mu\nu}, J_{\rho\sigma}] = \delta_{\mu\sigma}J_{\rho\nu} - \delta_{\nu\rho}J_{\mu\sigma} \tag{34}$$

These are the commutation relations of the Lie algebra associated with the general linear group, of which Λ is an element.

For simplicity, we will restrict the subsequent steps of our analysis to the case of no magnetic field. With no magnetic field, $\pi^i_\mu = p^i_\mu$ and it is straight forward to check that the strain generator commutation relations are satisfied by

$$J_{\mu\nu} = -\frac{1}{2}\sum_{i=1}^{N}\{x^i_\mu, p^i_\nu\} \tag{35}$$

We can show that there is a simple relation between the stress tensor and the strain generator. We derive the stress tensor from the momentum density continuity equation, given by

$$\partial_t g_\mu + \partial_\nu \tau_{\nu\mu} = 0 \tag{36}$$

[8]As stated earlier, we ultimately neglect electron-electron interactions. However, the formalism developed in the present section is unaffected by the presence of interactions, so we include them to demonstrate the generality of the results.

where $\tau_{\mu\nu}$ is the stress tensor. Letting the momentum density be given by $\vec{g}(\vec{r}) = \frac{1}{2}\sum_{i=1}^{N}\{\vec{p}^i, \delta(\vec{r} - \vec{x}^i)\}$, and evaluating the Fourier transform of the continuity equation at long wavelengths, we find

$$\partial_t \left(\frac{1}{2} \sum_{i=1}^{N} \{p_\mu^i, 1 - iq_\nu x_\nu^i\} \right) + iq_\nu \tau_{\nu\mu}(\vec{q} = \vec{0}) = 0 \tag{37}$$

Using $\partial_t(\sum_{i=1}^{N} p_\mu^i) = 0$ by translational invariance/momentum conservation, we find

$$\tau_{\nu\mu}(\vec{q} = \vec{0}) = -\partial_t \left(-\frac{1}{2} \sum_{i=1}^{N} \{x_\nu^i, p_\mu^i\} \right) = -\partial_t J_{\nu\mu} = -i[H_0, J_{\nu\mu}] \tag{38}$$

where in the last step we use the Heisenberg equation of motion. We define $T_{\nu\mu} = \tau_{\nu\mu}(\vec{q} = \vec{0}) = \int d^3x \tau_{\nu\mu}(\vec{x})$ to be the integrated stress tensor. From $H_\Lambda(t) = e^{-i\lambda_{\mu\nu}J_{\mu\nu}} H_0 e^{i\lambda_{\mu\nu}J_{\mu\nu}}$, we see that an equivalent definition for $T_{\mu\nu}$ is thus

$$T_{\mu\nu} = -\frac{\partial H_\Lambda}{\partial \lambda_{\mu\nu}}\bigg|_{\lambda_{\mu\nu}=0} \tag{39}$$

This is similar to the usual result from general relativity which expresses the stress tensor as a variation of the Hamiltonian with respect to the metric. While we have thus far worked with $T_{\mu\nu}$, the unperturbed integrated stress tensor, we may generalize equation (39) to define the perturbed integrated stress tensor. Namely:

$$T_{\mu\nu}^\Lambda = -\Lambda_{\rho\mu}\Lambda_{\nu\sigma}^{-1}\frac{\partial H_\Lambda}{\partial \lambda_{\rho\sigma}} \tag{40}$$

Just as the perturbed current density introduced in Section 4.1 is a function of the perturbation in the vector potential, the perturbed stress tensor is a function of the strain Λ. The extra factors of Λ and Λ^{-1} in the definition of the perturbed stress tensor may appear strange. We include them to simplify our results when we take a hydrodynamic view of viscosity. Intuitively, we think of viscosity as quantifying how the system responds to a dynamic rather than a static field, so we want our perturbing fields to be $\partial_t \lambda_{\mu\nu}$ rather than $\lambda_{\mu\nu}$. In that case, defining the perturbed stress tensor to instead be $T_{\mu\nu} = -\frac{\partial H}{\partial \lambda_{\mu\nu}}$ introduces undesirable contact terms proportional to $\lambda_{\mu\nu}$, as we shall see.[9]

Using equation (39), we write the strained Hamiltonian as $H_\Lambda = H_0 - T_{\mu\nu}\lambda_{\mu\nu} + O(\lambda^2)$. It may be tempting to declare $T_{\mu\nu}$ to be the perturbing operators and $\lambda_{\mu\nu}$ to be the perturbing functions. However, following the discussion in the previous paragraph, we use the product rule and instead write the perturbing Hamiltonian as

$$H_1 = -T_{\mu\nu}\lambda_{\mu\nu} = \partial_t J_{\mu\nu}\lambda_{\mu\nu} = \partial_t \left(J_{\mu\nu}\lambda_{\mu\nu}\right) - J_{\mu\nu}\partial_t \lambda_{\mu\nu} \tag{41}$$

Applying the formalism from Section 3, we have

$$\delta \left\langle T_{\mu\nu} \right\rangle (t) = -i \left\langle [T_{\mu\nu}(t), J_{\rho\sigma}(t)] \right\rangle_0 \lambda_{\rho\sigma}(t) - \int_{-\infty}^{\infty} dt' \eta_{\mu\nu\rho\sigma}(t - t') \frac{\partial \lambda_{\rho\sigma}(t')}{\partial t'} \tag{42}$$

where the response function $\eta_{\mu\nu\rho\sigma}$, which we define to be the viscosity, is

$$\eta_{\mu\nu\rho\sigma}(\omega) = -i \int_0^{\infty} dt e^{i\omega^+ t} \left\langle [T_{\mu\nu}(t), J_{\rho\sigma}(0)] \right\rangle_0 \tag{43}$$

[9]For the case of a homogeneous fluid, the two possible definitions of the perturbed stress tensor are equivalent. We should be careful, however, since the tilted field system we explore in Section 6 is an inhomogeneous fluid.

The first term in equation (42) comes from the total time-derivative term in equation (41). Conveniently, it cancels with another term that arises because what we're really interested in is the expectation value of the perturbed stress tensor, $T^{\Lambda}_{\mu\nu}$, which is not the same as the unperturbed stress tensor $T_{\mu\nu}$. From equations (39) and (40), we have $T^{\Lambda}_{\mu\nu} = T_{\mu\nu} + \lambda_{\rho\mu} T_{\rho\nu} - \lambda_{\nu\sigma} T_{\mu\sigma} - i[J_{\mu\nu}, T_{\rho\sigma}]\lambda_{\rho\sigma} + O(\lambda^2)$. Thus, the sum of the corrections to the expectation value of the perturbed stress tensor and the first term in equation (42) is

$$\Sigma = \lambda_{\rho\mu}(t) \langle T_{\rho\nu} \rangle (t) - \lambda_{\nu\sigma} \langle T_{\mu\sigma}(t) \rangle + i \langle [T_{\rho\sigma}, J_{\mu\nu}] - [T_{\mu\nu}, J_{\rho\sigma}] \rangle_0 \lambda_{\rho\sigma} \tag{44}$$

Using the Jacobi identity, the result $T_{\mu\nu} = -i[H_0, J_{\mu\nu}]$ and equation (34), we find

$$\begin{aligned} [T_{\rho\sigma}, J_{\mu\nu}] - [T_{\mu\nu}, J_{\rho\sigma}] &= -i \left([[H_0, J_{\rho\sigma}], J_{\mu\nu}] - [[H_0, J_{\mu\nu}], J_{\rho\sigma}] \right) \\ &= i[[J_{\rho\sigma}, J_{\mu\nu}], H_0] = [\delta_{\nu\rho} J_{\mu\sigma} - \delta_{\mu\sigma} J_{\rho\nu}, H_0] \\ &= i(\delta_{\mu\sigma} T_{\rho\nu} - \delta_{\nu\rho} T_{\mu\sigma}) \end{aligned} \tag{45}$$

It is clear therefore that $\Sigma = 0$. It is this cancellation which primarily motivates defining $T^{\Lambda}_{\mu\nu}$ via equation (40). A more relevant version of equation (42) is therefore

$$\langle T^{\Lambda}_{\mu\nu}(t) \rangle - \langle T_{\mu\nu}(t) \rangle_0 = -\int_{-\infty}^{\infty} dt' \eta_{\mu\nu\rho\sigma}(t - t') \frac{\partial \lambda_{\rho\sigma}(t')}{\partial t'} \tag{46}$$

with $\eta_{\mu\nu\rho\sigma}(\omega)$ still given in equation (43). This is the result we sought.

Finally, using $T_{\mu\nu} = -\partial_t J_{\mu\nu}$, time-translation invariance and integration by parts, we can re-express the viscosity in a strain-strain form or a stress-stress form. The latter is more useful to us, and takes the form

$$\begin{aligned} \eta_{\mu\nu\rho\sigma}(\omega) &= -i \int_0^{\infty} dt \frac{1}{i\omega^+} \partial_t e^{i\omega^+ t} \langle [T_{\mu\nu}(0), J_{\rho\sigma}(-t)] \rangle_0 \\ &= \frac{1}{\omega^+} \left[\langle T_{\mu\nu}(0), J_{\rho\sigma}(0) \rangle_0 + \int_0^{\infty} dt e^{i\omega^+ t} \langle [T_{\mu\nu}(0), \partial_t J_{\rho\sigma}(-t)] \rangle_0 \right] \\ &= \frac{1}{\omega^+} \left[\langle T_{\mu\nu}(0), J_{\rho\sigma}(0) \rangle_0 + \int_0^{\infty} dt e^{i\omega^+ t} \langle [T_{\mu\nu}(t), T_{\rho\sigma}(0)] \rangle_0 \right] \end{aligned} \tag{47}$$

Although we simplified our analysis by assuming that there is no magnetic field, the form of the major results in this section continue to hold when a magnetic field is introduced (see [2], [3]). In particular, we still have $T_{\mu\nu} = -i[H_0, J_{\mu\nu}]$, $H_1 = -J_{\mu\nu} \frac{\partial \lambda_{\mu\nu}}{\partial t}$, and the viscosity is still given by equation (47). The differences are in the definitions the strain generators, which require some care; for instance, for dilations, the magnetic field must be changed to keep the magnetic flux–and therefore the filling factor–fixed (see Section 2.3 of [3]). However, when we analyze the band mass and tilted field systems in Sections 5 and 6, we use the stress-stress form of the viscosity exclusively. We derive the forms of the stress tensors from the momentum density continuity equation, and thus do not directly work with the strain generators.

Before we conclude this subsection, it is worth discussing a subtle issue involving our viscosity results. Given $T_{\mu\nu} = -i[H_0, J_{\mu\nu}]$, one might conclude that $\langle T_{\mu\nu} \rangle_0 = -iE_0(\langle J_{\mu\nu} \rangle_0 - \langle J_{\mu\nu} \rangle_0) = 0$. On the other hand, we know that a homogeneous fluid, like a 2DEG in a uniform magnetic field, has pressure and its ground state stress tensor is of the form $\langle T_{\mu\nu} \rangle_0 = P\delta_{\mu\nu}$. The discrepancy is resolved by remembering that the Hamiltonian given in equation (32), in the case of repulsive interactions and no magnetic field, has no normalizable ground state. Evaluating expectation values in unnormalizable states can lead to unsound results.[10] Thus, we must either add a confining

[10] A classic example is $\langle p | [x, p] | p \rangle = 0 = i\delta(0)$. The absurd result is explained by $|p\rangle$ being unnormalizable.

potential to the Hamiltonian to make the energy eigenstates normalizable (in which case the continuity equation defining the stress tensor has an additional term) or we must evaluate expectation values in normalizable superpositions of the energy eigenstates, which are not energy eigenstates. In either case, generally $\langle T_{\mu\nu}\rangle \neq 0$.

In our case, the system features a magnetic field, which naively seems to make the ground states normalizable (i.e., an electron moving in two dimensions in a uniform magnetic field has a bounded orbit). However, since the magnetic field must be changed for dilations in the strain formalism, states corresponding to a fixed magnetic field are not eigenstates of the Hamiltonian. Therefore, working with states with fixed magnetic field means working with a superposition of eigenstates, so once again $\langle T_{\mu\nu}\rangle \neq 0$. Practically speaking, what this all means is that we must be careful simplifying terms of the form $\langle [A, H_0]\rangle_0$, but may otherwise proceed with the formalism we have introduced. In particular, if we define the stress tensor via the continuity equation rather than in terms of the strain generators, we do not need to directly worry about the issues that we've highlighted when we analyze the band mass anisotropy and tilted field systems.

4.3 Hall conductivity and Hall viscosity

We have derived expressions for the conductivity and viscosity of a quantum mechanical system. In this section we focus on the Hall conductivity and Hall viscosity. We define the Hall conductivity to be the antisymmetric part of the conductivity at $\omega = 0$:

$$\sigma^H_{\mu\nu} = \frac{1}{2}\lim_{\omega_+\to 0}(\sigma_{\mu\nu}(\omega) - \sigma_{\nu\mu}(\omega)) \tag{48}$$

Likewise, we define the Hall viscosity to be the antisymmetric part of the viscosity at $\omega = 0$:

$$\eta^H_{\mu\nu\rho\sigma} = \frac{1}{2}\lim_{\omega_+\to 0}(\eta_{\mu\nu\rho\sigma}(\omega) - \eta_{\rho\sigma\mu\nu}(\omega)) \tag{49}$$

Let us examine the contact term in equation (47). Using the calculation in equation (45), we find that the antisymmetric part of the contact term is given by

$$\frac{1}{\omega^+}\langle [T_{\mu\nu}, J_{\rho\sigma}] - [T_{\rho\sigma}, J_{\mu\nu}]\rangle_0 = \frac{i}{\omega^+}(\delta_{\nu\rho}\langle T_{\mu\sigma}\rangle_0 - \delta_{\mu\sigma}\langle T_{\rho\nu}\rangle_0) \tag{50}$$

For $\langle T_{\mu\nu}\rangle = P\delta_{\mu\nu}$, which is the form of the stress tensor of a homogeneous fluid, the antisymmetric part of the contact term vanishes. We will therefore be able to ignore the contact term when calculating the Hall viscosity for the system with band mass anisotropy. However, the contact term does not vanish for the tilted field system, and instead reflects an interesting physical property. Notice that if the antisymmetric part of the contact term is non-zero, then it diverges as $\omega^+ \to 0$. We will therefore analyze it separately from the rest of the Hall viscosity.

Recalling that $A(t) = e^{iHt}Ae^{-iHt}$ for generic operator A, we can insert a complete set of energy eigenstates and evaluate the integral over time in the definition of the response functions to derive two alternative definitions for the Hall conductivity and the (sans contact term) Hall viscosity:

$$\sigma^H_{\mu\nu} = 2\sum_{\beta\neq 0}\frac{\mathrm{Im}(\langle 0|J_\mu|\beta\rangle\langle\beta|J_\nu|0\rangle)}{(E_0 - E_\beta)^2} \tag{51}$$

$$\eta^H_{\mu\nu\rho\sigma} = 2\sum_{\beta\neq 0}\frac{\mathrm{Im}\left(\langle 0|T_{\mu\nu}|\beta\rangle\langle\beta|T_{\rho\sigma}|0\rangle\right)}{(E_0 - E_\beta)^2} \tag{52}$$

The operators are now evaluated in the Schrödinger picture. In these expressions β is a label for the complete set of energy states and 0 is the ground state, assumed to be unique. Although the Hamiltonians we consider are highly degenerate, we will see that the matrix elements of the J and T operators depend only on the eigenspaces, not on the specific states within the eigenspaces used to evaluate them. Therefore, by restricting β to refer to only one state per energy eigenspace, the formulas above apply to the calculations that interest us. Note also that equation (52) makes the dimensions of the integrated Hall viscosity very clear: since $T \sim \frac{\partial H}{\partial \lambda}$ has units of energy, the integrated Hall viscosity is dimensionless.

Also useful is the fact that, in the usual case where the stress tensor operator is symmetric, we have $\eta^H_{\mu\nu\rho\sigma} = \eta^H_{\nu\mu\rho\sigma} = \eta^H_{\mu\nu\sigma\rho} = -\eta^H_{\rho\sigma\mu\nu}$.[11] We can then deduce that η^H has $n(d) = d(d+1)(d(d+1)-2)/8$ independent components. For the two cases that interest us, we have $n(2) = 3$ and $n(3) = 15$.

Finally, for 2D systems, we define the so-called contracted Hall viscosity to be the following two-index tensor:

$$\eta^H_{ab} = \frac{1}{2}\epsilon_{ac}\epsilon_{bd}\epsilon_{ef}\eta^H_{cedf} \tag{53}$$

Introduced by Haldane (see, e.g., [8]), the contracted Hall tensor is the form of the viscosity investigated in [6], the paper which most directly motivates our research. We will use the contracted Hall tensor and the results from [6] to derive an effective anisotropy matrix for the tilted field system in Section 7. Note, if the stress tensor operator is symmetric, then η^H_{ab} is symmetric as well, and therefore has 3 independent components. Thus, the contracted Hall tensor usually fully captures the behavior of the Hall viscosity in two dimensions, thereby justifying its use.

5 IQHE with band mass anisotropy

We first analyze the system with band mass anisotropy, whose Hamiltonian is given in equation (2). It is a bona fide 2D system with anisotropy introduced in a particularly simple way. The results derived here will provide context for the results derived for the tilted field system. Additionally, the case $\tilde{m}_{ab} = \frac{1}{m}\delta_{ab}$ automatically gives the results for the isotropic system.

For simplicity, we consider the single particle (i.e., $N = 1$) form of the Hamiltonian defined in equation (2). Once we have diagonalized the Hamiltonian, and calculated the Hall conductivity, stress tensor, and Hall viscosity, it is straightforward to determine the results for a multiparticle system. Because we are neglecting interactions, the observables scale linearly with the number of particles in the ground state.

5.1 Solving the band mass eigenproblem with ladder operators

We first diagonalize the single particle Hamiltonian, following [14]. We start with a mass tensor m_{ab} and its inverse, $\tilde{m}_{ab}$, which satisfy $m_{ab}\tilde{m}_{bc} = \delta_{ac}$. We can decompose m_{ab} and $\tilde{m}_{ab}$ in terms of complex vectors $\vec{\omega} = (\omega_x, \omega_y)$ and $\vec{\nu} = (\nu_x, \nu_y)$, writing $m_{ab} = m(\omega_a^*\omega_b + \omega_b^*\omega_a)$ and $\tilde{m}_{ab} = \frac{1}{m}(\nu_a^*\nu_b + \nu_b^*\nu_a)$. In terms of the components of the matrices,

$$(\omega_x, \omega_y) = \frac{e^{i\phi}}{\sqrt{2m}}\left(\sqrt{m_{11}}, \sqrt{m_{22}}e^{i\alpha}\right) \tag{54}$$

$$(\nu_x, \nu_y) = \frac{e^{i\theta}}{2i\mathrm{Im}(\omega_x\omega_y^*)}(-\omega_y, \omega_x) \tag{55}$$

[11] Note that while the contact term is antisymmetric under the double exchange of indices, it is not symmetric under a single exchange of the first two or latter two indices.

where $\alpha = \arccos \frac{m_{12}}{\sqrt{m_{11}m_{22}}}$. The parameters m, ϕ and θ are redundancies in the description and we use the freedom to set $\theta = \phi = 0$ and $m^2 = \det(m_{ab})$. We can then write down some useful, interrelated identities:

$$\mathrm{Im}(\omega_a\omega_b^*) = \mathrm{Im}(\nu_a\nu_b^*) = -\frac{1}{2}\epsilon_{ab}, \qquad \nu_a = -i\epsilon_{ab}\omega_b \tag{56}$$

$$\mathrm{Re}(\omega_a\nu_b^*) = \frac{1}{2}\delta_{ab} \qquad \omega_a = -i\epsilon_{ab}\nu_b \tag{57}$$

$$\omega_a\nu_a = -i\epsilon_{ab}\omega_a\omega_b = -i\epsilon_{ab}\nu_a\nu_b = 0 \qquad \omega_a\nu_a^* = i\epsilon_{ab}\omega_a\omega_b^* = i\epsilon_{ab}\nu_a\nu_b^* = 1 \tag{58}$$

The Hamiltonian becomes

$$H = \frac{1}{2m}\left(\nu_a^*\nu_b + \nu_a\nu_b^*\right)\pi_a\pi_b \tag{59}$$

and if we define $b = \frac{1}{\sqrt{B}}\nu_a^*\pi_a$, which satisfies $[b, b^\dagger] = \frac{1}{B}\nu_a^*\nu_b[\pi_a, \pi_b] = -i\epsilon_{ab}\nu_a^*\nu_b = 1$, we get

$$H = \frac{\omega_c}{2}(bb^\dagger + b^\dagger b) = \omega_c\left(b^\dagger b + \frac{1}{2}\right) \tag{60}$$

where $\omega_c \equiv \frac{B}{m}$ is the cyclotron frequency; it should not be confused with the c-component of $\vec{\omega}$.

We have successfully diagonalized the Hamiltonian in terms of the raising/lowering operators $(b^\dagger, b)$. A second lowering operator is given by $a = \sqrt{B}\omega_a R_a$, where $R_a = x_a - \frac{\epsilon_{ab}}{B}\pi_b$ is commonly called the guiding center coordinate. It is simple to check that $[R_a, x_b] = \frac{i}{B}\epsilon_{ab}$, $[R_a, \pi_b] = 0$ and $[R_a, R_b] = \frac{i}{B}\epsilon_{ab}$, from which we confirm $[a, a^\dagger] = 1$ and $[a, b] = [a, b^\dagger] = 0$. We can therefore immediately write-down the generic energy eigenstate as

$$|n, m\rangle = \frac{(a^\dagger)^m(b^\dagger)^n}{\sqrt{m!n!}}|0\rangle \tag{61}$$

where the ground state $|0\rangle$ is annihilated by a and b. Finally, the momenta can be expressed simply in terms of the $(b^\dagger, b)$ operators:

$$\pi_a = -i\sqrt{B}\epsilon_{ab}\left(\nu_b b - \nu_b^* b^\dagger\right) = \sqrt{B}(\omega_a b + \omega_a^* b^\dagger) \tag{62}$$

5.2 Hall conductivity

The Hall conductivity for the band mass anisotropy is simple to calculate. We first need to determine the integrated current operators, and then apply equations (31) and (48). The definition of the current density as the derivative of the action with respect to the vector potential, along with some dimensional analysis, tells us that the integrated current density is given by $J_a = -\tilde{m}_{ab}\pi_b$. Notice that for $\tilde{m}_{ab} = \frac{1}{m}\delta_{ab}$, $J_a = -\dot{x}_a$. Since $e = 1$, this is what we classically expect.

From equations (60) and (62), we deduce the time-dependence of $\pi_a(t)$ using the fact that raising/lowering operators evolve particularly simply (i.e., from $[H, b] = -i\omega_c b$ and Heisenberg's equation of motion, we get $b(t) = e^{-i\omega_c}b(0)$). The conductivity is thus given by

$$\begin{aligned}
\sigma_{ab}(\omega) &= \frac{\tilde{m}_{ac}\tilde{m}_{bd}\epsilon_{ce}\epsilon_{df}}{\omega^+}\int_0^\infty dt e^{i\omega^+ t}\left\langle[\pi_e(t), \pi_f(0)]\right\rangle_0 \\
&= -\frac{\tilde{m}_{ac}\tilde{m}_{bd}\epsilon_{ce}\epsilon_{df}B}{\omega^+}\int_0^\infty dt e^{i\omega^+ t}\left\langle[\nu_e e^{-i\omega_c t}b(0) - \nu_e^* e^{i\omega_c t}b^\dagger(0), \nu_f b(0) - \nu_f^* b^\dagger(0)]\right\rangle_0 \\
&= iB\tilde{m}_{ac}\tilde{m}_{bd}\epsilon_{ce}\epsilon_{df}\frac{1}{\omega^+}\left[\frac{\nu_e^*\nu_f}{\omega^+ + \omega_c} - \frac{\nu_e\nu_f^*}{\omega^+ - \omega_c}\right]
\end{aligned} \tag{63}$$

After a string of simplifications using the properties of $\vec{\omega}$ and $\vec{\nu}$, the Hall conductivity reduces to

$$\sigma^H_{ab} = \frac{1}{2}\lim_{\omega^+\to 0}(\sigma_{ab}(\omega) - \sigma_{ba}(\omega)) = -\frac{i}{B}m^2\tilde{m}_{ac}\tilde{m}_{bd}\epsilon_{ce}\epsilon_{df}[\nu^*_e\nu_f - \nu^*_f\nu_e] = \frac{1}{B}\epsilon_{ab} \tag{64}$$

If we re-insert the electron charge e explicitly, include the contributions from all N electrons and divide by the area to get the intrinsic rather than the integrated conductivity, we find

$$\sigma^H_{ab} = \frac{e\bar{n}}{B}\epsilon_{ab} \tag{65}$$

where $\bar{n} \equiv \frac{N}{A}$ is the electron density. It is the usual result for the isotropic system (see, e.g., [18]). What is surprising, however, is that the conductivity does not depend explicitly on the mass tensor m_{ab}. Moreover, if we use the rule of thumb that there is one electron state per area containing one quantum of magnetic flux per energy level, the ground state density is also independent of the anisotropy because the anisotropy affects neither the size of the system nor the magnetic field.[12] Thus, the conductivity in the ground state is anisotropy independent.

5.3 Stress-energy tensor

In this section we derive the form of the integrated stress tensor. We seek an operator τ_{ab} that in the Heisenberg picture obeys the momentum continuity equation:

$$\frac{\partial g_a}{\partial t} + \partial_b\tau_{ba} = f^L_a \tag{66}$$

where the Lorentz force density is given by $f^L_a = B\epsilon_{ab}j_b = -B\epsilon_{ab}\tilde{m}_{bc}g_c$. From $g_a(\vec{r}) = 1/2\{\pi_a, \delta(\vec{r}-\vec{x})\}$ for a single particle, we have

$$\begin{aligned}\frac{\partial g_a}{\partial t} = i[H, g_a] &= \frac{i}{4}\tilde{m}_{bc}[\pi_b\pi_c, \{\pi_a, \delta(\vec{r}-\vec{x})\}] \\ &= \frac{i}{4}\tilde{m}_{bc}\left(\{[\pi_b\pi_c, \pi_a], \delta(\vec{r}-\vec{x})\} + \{[\pi_b\pi_c, \delta(\vec{r}-\vec{x})], \pi_a\}\right)\end{aligned} \tag{67}$$

Given $[\pi_b, \pi_c] = -iB\epsilon_{bc}$ and $\tilde{m}_{bc} = \tilde{m}_{cb}$, the first term in the parentheses becomes

$$\tilde{m}_{bc}\{[\pi_b\pi_c, \pi_a], \delta(\vec{r}-\vec{x})\} = -2iB\tilde{m}_{bc}\epsilon_{ba}\{\pi_c, \delta(\vec{r}-\vec{x})\} = -4iB\tilde{m}_{bc}\epsilon_{ba}g_c(\vec{r}) \tag{68}$$

Using $[p_a, f(x)] = -i\frac{\partial f}{\partial x_a}$, the second term is

$$\tilde{m}_{bc}\{[\pi_b\pi_c, \delta(\vec{r}-\vec{x})], \pi_a\} = -2i\tilde{m}_{bc}\{\pi_b, \pi_a\}\frac{\partial}{\partial x_c}\delta(\vec{r}-\vec{x}) = 2i\tilde{m}_{bc}\{\pi_b, \pi_a\}\frac{\partial}{\partial r_c}\delta(\vec{r}-\vec{x}) \tag{69}$$

Thus we have

$$\frac{\partial g_a}{\partial t} = -B\tilde{m}_{bc}\epsilon_{ab}g_c - \frac{\partial}{\partial r_c}\left(\frac{1}{2}\tilde{m}_{bc}\{\pi_b, \pi_a\}\delta(\vec{r}-\vec{x})\right) \tag{70}$$

[12] There is a beautiful argument presented in [7] that uses the magnetic translation group and properties of analytic functions to make the one-state-per-flux-quantum rule rigorous. Since the complex plane is two dimensional, the result applies to our 2D band mass anisotropy system, but not to our 3D tilted field system. To demonstrate that the ground state degeneracy in the tilted field system does not depend on the tilt, we will use more plebeian methods.

The first term is f_a^L, so we recognize $\tau_{ab} = \frac{1}{2}\tilde{m}_{ac}\{\pi_b, \pi_c\}\delta(\vec{r} - \vec{x})$. Therefore, the integrated stress tensor is

$$T_{ab} = \int d^2 r \tau_{ab}(\vec{r}) = \sum_{i=1}^{N} \frac{1}{2}\tilde{m}_{ac}\{\pi_b^i, \pi_c^i\} \tag{71}$$

where in the last step we included the contributions from all N electrons. If $\tilde{m}_{ab} = \frac{1}{m}\delta_{ab}$, we get the usual result for the isotropic stress tensor, $T_{ab} = \frac{1}{2m}\sum_i \{\pi_a^i, \pi_b^i\}$.

Since we have solved for the energy eigenstates and derived the form of the stress-energy tensor, it is straightforward to compute the expectation value of the stress tensor in the ground state:

$$\begin{aligned} \langle T_{ab} \rangle &= \frac{1}{2}\tilde{m}_{ac} \langle 0|\{\pi_c, \pi_b\}|0\rangle = -\frac{B}{2}\tilde{m}_{ac}\epsilon_{ce}\epsilon_{bd} \langle 0|\{\nu_e b - \nu_e^* b^\dagger, \nu_d b - \nu_d^* b^\dagger\}|0\rangle \\ &= \frac{B}{2}\tilde{m}_{ac}\epsilon_{ce}\epsilon_{bd} (\nu_e^* \nu_d + \nu_d^* \nu_e) = \frac{B}{2m}\tilde{m}_{ac} m_{cb} = \frac{\omega_c}{2}\delta_{ab} \end{aligned} \tag{72}$$

If we re-insert factors of e and $\hbar$, include the contributions of all N particles and divide by the area to get the intrinsic stress tensor, we find

$$\langle \tau_{ab} \rangle = \frac{\hbar \omega_c \bar{n}}{2}\delta_{ab} \tag{73}$$

This result is also independent of the anisotropy. It is the stress tensor for a homogeneous fluid.

5.4 Hall viscosity

To compute the Hall viscosity, it's probably simplest to use equation (52). After significant simplification, we find

$$\begin{aligned} \eta_{abcd}^H &= \frac{1}{2}\tilde{m}_{ae}\tilde{m}_{cf} \sum_{\beta \neq 0} \frac{\mathrm{Im}\left(\langle 0|\{\pi_b, \pi_e\}|\beta\rangle \langle \beta|\{\pi_d, \pi_f\}|0\rangle\right)}{(\omega_c \beta)^2} \\ &= m^2 \tilde{m}_{ae}\tilde{m}_{cf}\epsilon_{bg}\epsilon_{eh}\epsilon_{di}\epsilon_{fj}\mathrm{Im}\left(\nu_g \nu_h \nu_i^* \nu_j^*\right) \\ &= \bar{n}\mathrm{Im}\left(\nu_a \omega_b \nu_c^* \omega_d^*\right) \end{aligned} \tag{74}$$

In the last step we explicitly included the contributions from all N particles and dividing by the area to get the intrinsic viscosity. Equation (74) can be simplified further if we use the fact that $\tilde{m}_{ab}$ is real symmetric and therefore diagonalizable. Without loss of generality, let $m_{ab} = m \ \mathrm{diag}(\alpha, 1/\alpha)$. It is easy to check that in that case $\vec{\omega} = \frac{1}{\sqrt{2}}(\sqrt{\alpha}, \frac{i}{\sqrt{\alpha}})$, $\vec{\nu} = \frac{1}{\sqrt{2}}(\frac{1}{\sqrt{\alpha}}, \sqrt{\alpha} i)$ and therefore $\omega_a \nu_b^* = \{\{1, -i\alpha\}, \{i/\alpha, 1\}\} = \frac{1}{2}(\delta_{ab} - i\epsilon_{ac}(m\tilde{m}_{bc}))$. Although we assumed we were in the eigenframe to arrive at this identity, it is expressed using only tensors and thus holds in any frame. Therefore, the Hall viscosity is generally given by

$$\eta_{abcd}^H = -\frac{\bar{n}}{4}\left[\delta_{ad}\epsilon_{be}(m\tilde{m}_{ce}) + \delta_{bc}\epsilon_{ed}(m\tilde{m}_{ae})\right] \tag{75}$$

For the case $\tilde{m}_{ab} = \frac{1}{m}\delta_{ab}$, this reduces to the known result for the isotropic Hamiltonian (see, e.g., [3]), which is alternatively often derived using symmetry arguments. That is, there is a unique rotationally symmetric 2-component 2D tensor that is symmetric under the interchange of the first and second indices, and antisymmetric under the interchange of the first two and latter two indices: $\delta_{ad}\epsilon_{bc} + \delta_{bc}\epsilon_{ad}$. To get the proportionality constant, one has to carry out a more precise calculation, as we did.

We can also straightforwardly compute the contracted Hall viscosity:

$$\eta^H_{ab} = \frac{\bar{n}}{2}\epsilon_{ac}\epsilon_{bd}\epsilon_{ef}\mathrm{Im}\left(\nu_c\omega_e\nu_d^*\omega_f^*\right) = -\frac{\bar{n}}{2}\mathrm{Im}\left(i\omega_a\omega_b^*\right) = -\frac{\bar{n}}{4}\left(\frac{1}{m}m_{ab}\right) \tag{76}$$

This agrees with the result determined in [6] up to a difference in sign, which merely reflects a difference in charge convention.

6 IQHE with tilted field anisotropy

In this section, we finally examine the tilted field system described by the Hamiltonian given in equation (3). To reiterate, we put the electrons in a harmonic potential $V(z) = \frac{1}{2}m\omega_0^2 z^2$ and a magnetic field $\vec{B} = B_x\hat{x} + B_z\hat{z}$. We introduce cyclotron frequencies $\omega_x \equiv B_x/m$, $\omega_z \equiv B_z/m$, and dimensionless parameters $l \equiv \frac{\omega_z}{\omega_0}$, $k \equiv \frac{\omega_x}{\omega_z}$. We will consider the case of a single electron; since we neglect interactions, all observables scale linearly with the number of electrons in the ground state. Furthermore, we are interested in the regime of weak tilt and strong confinement, so we ultimately express our results to leading or sub-leading order in l and k.

We first characterize the energy eigenstates and then compute the Hall conductivity, stress tensor, and Hall viscosity. We analyze not only the ground state expectation value of the stress tensor, but also examine how the operators look when projected into the plane. We find that all three observables have interesting properties, indicating that the problem is more subtle than the simple set-up would suggest.

6.1 Solving the tilted field eigenproblem

We begin by diagonalizing the Hamiltonian in two ways. First, we pick a convenient Landau-like gauge and work with the resulting wavefunctions. Second, we use ladder operators to determine the energy eigenstates in a gauge invariant way. Both approaches have advantages. The ladder operator approach is conceptually clean and the results hold for all gauge choices. Meanwhile, the fixed-gauge approach is at times easier to calculate with, allows us to interpret the ladder operators, and makes determining the degeneracy of the ground state particularly simple.[13]

6.1.1 Using a Landau-type gauge

In this section, we solve for the energy eigenstates by picking a gauge. Our approach is similar to the one taken in [13]. Let the vector potential be $\vec{A} = (0, B_z x - B_x z, 0)$ so that $\vec{B} = (B_x, 0, B_z)$, as desired. The Hamiltonian becomes

$$H = \frac{1}{2m}\left(p_x^2 + (p_y + B_z x - B_x z)^2 + p_z^2\right) + \frac{1}{2}m\omega_0^2 z^2 \tag{77}$$

Since $[H, p_y] = 0$, we may look for energy eigenstates $|\psi, q\rangle$ satisfying $p_y|\psi, q\rangle = q|\psi, q\rangle$. Within a q eigenspace, $H_q = \frac{1}{2m}\left(p_x^2 + p_z^2 + (q + B_z x - B_x z)^2\right) + \frac{1}{2}m\omega_0^2 z^2$. Next, we shift frames so that $x' = x + \frac{q}{B_z}$. Notice $[x', z] = [x', p_z] = 0$, $[x', p_x] = i$. Substituting and expanding, we find

$$H_q = \frac{1}{2m}\left(p_x^2 + p_z^2\right) + \frac{1}{2}m\omega_z^2 x'^2 + \frac{1}{2}m(\omega_x^2 + \omega_0^2)z^2 - m\omega_x\omega_z x' z \tag{78}$$

[13] We also carried out most of our calculations using both methods to check our results.

Up to a rotation, this is the Hamiltonian of an anisotropic 2D harmonic oscillator. To effect the rotation, let $A = \omega_z^2$, $B = \omega_x^2 + \omega_0^2$ and $C = 2\omega_x\omega_z$. Let α be defined by $\tan 2\alpha = \frac{C}{A-B}$, and introduce rotated positions and momenta u, v, p_u and p_v given by

$$\begin{pmatrix} u \\ v \end{pmatrix} = \begin{pmatrix} \cos\alpha & -\sin\alpha \\ \sin\alpha & \cos\alpha \end{pmatrix} \begin{pmatrix} x' \\ z \end{pmatrix} \iff \begin{pmatrix} x' \\ z \end{pmatrix} = \begin{pmatrix} \cos\alpha & \sin\alpha \\ -\sin\alpha & \cos\alpha \end{pmatrix} \begin{pmatrix} u \\ v \end{pmatrix} \tag{79}$$

$$\begin{pmatrix} p_u \\ p_v \end{pmatrix} = \begin{pmatrix} \cos\alpha & -\sin\alpha \\ \sin\alpha & \cos\alpha \end{pmatrix} \begin{pmatrix} p_x \\ p_z \end{pmatrix} \iff \begin{pmatrix} p_x \\ p_z \end{pmatrix} = \begin{pmatrix} \cos\alpha & \sin\alpha \\ -\sin\alpha & \cos\alpha \end{pmatrix} \begin{pmatrix} p_u \\ p_v \end{pmatrix} \tag{80}$$

Then, upon substitution of the new coordinates into H_q, we find

$$H_q = \frac{1}{2m}p_u^2 + \frac{1}{2}m\omega_u^2 u^2 + \frac{1}{2m}p_v^2 + \frac{1}{2}m\omega_v^2 v^2 \tag{81}$$

where $\omega_u^2 = A\cos\alpha^2 + B\sin\alpha^2 + \frac{C}{2}\sin 2\alpha$ and $\omega_v^2 = A\sin\alpha^2 + B\cos\alpha^2 - \frac{C}{2}\sin 2\alpha$. We may introduce lowering operators $a_u = \frac{1}{\sqrt{2m}}(mu + ip_u)$ and $a_v = \frac{1}{\sqrt{2m}}(mv + ip_v)$. Then the Hamiltonian becomes $H_q = \omega_u(a_u^\dagger a_u + 1/2) + \omega_v(a_v^\dagger a_v + 1/2)$ and we can write the energy eigenstates as $|n_u, n_v, q\rangle$ such that $p_y |n_u, n_v, q\rangle = q |n_u, n_v, q\rangle$, $a_u |n_u, n_v, q\rangle = \sqrt{n_u} |n_u - 1, n_v, q\rangle$, $a_u^\dagger |n_u, n_v, q\rangle = \sqrt{n_u + 1} |n_u, n_v, q\rangle$ and likewise for a_v and $a_v^\dagger$.

Actually, we want to make a slight modification. The operators a_u and a_v that we introduced depend on q because u and v depend on x', which depends on q. It is better to have operators a_u and a_v that are defined without reference to a particular eigenspace of p_y, and to put the q-dependence into the eigenstates. This can be done by replacing x' by x in equation (79) and rewriting the energy eigenstates as

$$|\psi\rangle = Q\left(-\frac{q}{B_z}\hat{x}\right)|n_u, n_v, q\rangle, \qquad \text{with} \qquad H|\psi\rangle = (\omega_u(n_u + 1/2) + \omega_v(n_v + 1/2))|\psi\rangle \tag{82}$$

where $Q(\vec{a}) = e^{-i\vec{p}\cdot\vec{a}}$ is the translation operator. Thus, having momentum in the y-direction corresponds to a shift in the x-direction.[14] The states $|n_u, n_v, q\rangle$ and raising/lowering operators $(a_u^\dagger, a_u)$, $(a_v^\dagger, a_v)$ still satisfy the relations given in the paragraph below equation (81), but they are new, distinct states and operators.

What have we accomplished? Firstly, we have found the spectrum for the Hamiltonian:

$$E_{n_u,n_v,q} = \omega_u(n_u + 1/2) + \omega_v(n_v + 1/2) \tag{83}$$

We note that the energies are independent of q, so the Hamiltonian is highly degenerate, as in the isotropic case. We have also written down the energy eigenstates, and can therefore calculate expectation values of operators by writing the operators in terms of y, p_y, a_u, $a_u^\dagger$, a_v and $a_v^\dagger$. We can write down the form of the wavefunctions explicitly:

$$\psi_{m,n,q}(x,y,z) = \frac{e^{iqy}}{\sqrt{2\pi}} f_m\left(\left(x + \frac{q}{B_z}\right)\cos\alpha - z\sin\alpha; \omega_u\right) f_n\left(\left(x + \frac{q}{B_z}\right)\sin\alpha + z\cos\alpha; \omega_v\right) \tag{84}$$

where $f_n(x;\omega)$ is the nth eigenfunction of the simple harmonic oscillator with frequency ω (see [5]):

$$f_n(x;\omega) = \left(\frac{m\omega}{\pi}\right)^{1/4} \frac{1}{\sqrt{2^n n!}} H_n\left(\sqrt{m\omega}x\right) e^{-\frac{m\omega x^2}{2}} \tag{85}$$

[14]This statement shouldn't be interpreted too literally, since q is not gauge invariant.

$H_n(x)$ is the nth Hermitian polynomial. Note, we should not attach too much significance to the properties of the wavefunctions. For instance, it has drastically different dependence on x and y, even though the two coordinates are similar (and in the case $B_x = 0$, are physically identical), which is a result of the degeneracy and our gauge choice rather than a physical effect.

A few comments: first, as ω_x tends to 0, α tends to 0, and the x-z axes and u-v axes coincide. Thus, for a small parallel magnetic field component, $(a_u^\dagger, a_u)$ can be interpreted as raising/lowering operators for the in-plane Landau levels while $(a_v^\dagger, a_v)$ are raising/lowering operators along the z-axis. In the same limit, ω_u tends to ω_z and ω_v tends to ω_0. This comment will be useful in the next two sections when we project the stress-energy tensor into the plane using ladder operators.

Secondly, we can comment more precisely on the degeneracy of the different energy levels of the tilted field Hamiltonian. We use a heuristic argument similar to that in [13], [18]. Let us put our system in an $L_x \times L_y \times L_z$ box centered at the origin. The eigenfunctions described in equation (84) look like tubes running parallel to the y-axis and tilted in the x-z plane. They have a characteristic width of $1/\sqrt{m\omega_u}$ and characteristic height of $1/\sqrt{m\omega_v}$, which tend to $1/\sqrt{B_z}$ and $1/\sqrt{m\omega_0}$ in the small tilt limit. They are shifted in the x-direction depending on q. Because the system is finite, q is restricted to integer multiples of $\frac{2\pi}{L_y}$. Looking at equation (82) or (84) we see that if p_y is outside the range $[-L_xB_z/2, L_xB_z/2]$, then the center of the wavefunction is no longer in the box. Heuristically, we conclude that q only spans an interval of length L_xB_z, and therefore the degeneracy of the ground state is given by

$$\mathcal{N} = \frac{L_xB_z}{2\pi/L_y} = \frac{L_zL_yB_z}{2\pi} = \frac{AB_z}{\phi_0} \tag{86}$$

Thus, the degeneracy is equal to the flux of the perpendicular magnetic field through the x-y plane in units of the flux quantum. This result does not depend on the magnitude of the parallel component of the magnetic field, nor the strength of the confining potential.

6.1.2 Using gauge invariant ladder operators

In this section we diagonalize the Hamiltonian using ladder operators. We closely follow the derivation carried out in [20], with some computational details left out. We being by writing the Hamiltonian as

$$H = \frac{1}{2m}\left(\pi_x^2 + \pi_y^2 + \pi_z^2 + (m\omega_0 z)^2\right) \tag{87}$$

We then introduce the operators $a = \frac{1}{\sqrt{2B_z}}(\pi_x - i\pi_y)$ and $b = \frac{1}{\sqrt{2m\omega_0}}(\pi_y - im\omega_0 z)$ which satisfy $[a, a^\dagger] = [b, b^\dagger] = 1$ and $[a, b] = [a, b^\dagger] = -\frac{1}{2}\frac{B_x}{\sqrt{B_z m\omega_0}} = -\frac{1}{2}\frac{\omega_x^2}{\sqrt{\omega_z\omega_0}}$. The Hamiltonian becomes

$$H = \frac{\omega_z}{2}(a^\dagger a + aa^\dagger) + \frac{\omega_0}{2}(b^\dagger b + bb^\dagger) \tag{88}$$

Because a and b are not independent raising/lowering operators, the Hamiltonian is not yet diagonalized. We define a new operator α so that $\alpha = a + \frac{\omega_x}{2\sqrt{\omega_z\omega_0}}(b - b^\dagger) = a - \frac{i}{\sqrt{2B_z}}B_xz$ in which case $[\alpha, \alpha^\dagger] = 1$, $[\alpha, b] = [\alpha, b^\dagger] = 0$. Thus, α and b now have the right commutation relations, but the Hamiltonian is no longer diagonal:

$$\begin{aligned} H = &\frac{\omega_z}{2}\{\alpha, \alpha^\dagger\} + \frac{1}{2}\left(\omega_0 + \frac{\omega_x^2}{2\omega_0}\right)\{b, b^\dagger\} \\ &- \frac{\omega_x^2}{4\omega_0}((b^\dagger)^2 + b^2) + \frac{\omega_x}{2}\sqrt{\frac{\omega_z}{\omega_0}}(\alpha^\dagger b^\dagger + \alpha b - \alpha^\dagger b - \alpha b^\dagger) \end{aligned} \tag{89}$$

Next, we make use of a general procedure known as a Bogoliubov transformation. We want to introduce new ladder operators X and Y that diagonalize the Hamiltonian and are related to the α and b operators by a 4×4 matrix V and its inverse U:

$$[X^\dagger, Y^\dagger, X, Y]^T = V[b^\dagger, \alpha^\dagger, b, \alpha]^T \iff [b^\dagger, \alpha^\dagger, b, \alpha]^T = U[X^\dagger, Y^\dagger, X, Y] \tag{90}$$

To determine whether the Bogoliubov transformation is practicable, we can compute the so-called "dynamical matrix" $\mathcal{D}$ defined by $[H, (b^\dagger, \alpha^\dagger, b, \alpha)^T] = \mathcal{D}(b^\dagger, \alpha^\dagger, b, \alpha)^T$. This is a straightforward but messy computation, and it turns out that $\mathcal{D}$ can be diagonalized: $\mathcal{D} = U\Lambda U^{-1}$. Then $(X^\dagger, Y^\dagger, X, Y)^T = U^{-1}(b^\dagger, \alpha^\dagger, b, \alpha)^T$ has the correct ladder commutation relations, and obeys $[H, (X^\dagger, Y^\dagger, X, Y)^T] = \Lambda(X^\dagger, Y^\dagger, X, Y)^T$. Since each term in the Hamiltonian is quadratic in raising/lowering operators, we conclude that H only has terms of the form $XX^\dagger$, $X^\dagger X$, $YY^\dagger$ and $Y^\dagger Y$. Being careful about the details, Yang et al. get a Hamiltonian of the form

$$H = \omega_1(X^\dagger X + XX^\dagger) + \omega_2(Y^\dagger Y + YY^\dagger) \tag{91}$$

where, defining $\epsilon_1 = \omega_z^2 + \omega_0^2 + \omega_x^2$ and $\epsilon_2 = 2\omega_0\omega_z$, we have the oscillator frequencies

$$\omega_1^2 = \frac{1}{2}\left(\epsilon_1 - \sqrt{\epsilon_1^2 - \epsilon_2^2}\right) \tag{92}$$

$$\omega_2^2 = \frac{1}{2}\left(\epsilon_1 + \sqrt{\epsilon_1^2 - \epsilon_2^2}\right) \tag{93}$$

The explicit forms of U and V are given below. For ease of notation and without loss of generality, we set $\omega_z = 1$. Then:

$$V = \frac{1}{2\sqrt{\omega_2^2 - \omega_1^2}} \begin{pmatrix} U_1 & U_2 \\ U_2^* & U_1^* \end{pmatrix} \tag{94}$$

$$U = \frac{1}{2\sqrt{\omega_2^2 - \omega_1^2}} \begin{pmatrix} U_1^\dagger & -U_2^T \\ -U_2^\dagger & U_1^T \end{pmatrix} \tag{95}$$

where

$$U_1 = \begin{pmatrix} i(1+\omega_2)\sqrt{\frac{1-\omega_1^2}{\omega_2}} & i(1+\omega_1)\sqrt{\frac{\omega_2^2-1}{\omega_1}} \\ (1+\omega_1)\sqrt{\frac{\omega_2^2-1}{\omega_1}} & -(1+\omega_2)\sqrt{\frac{1-\omega_1^2}{\omega_2}} \end{pmatrix} \tag{96}$$

$$U_2 = \begin{pmatrix} i(\omega_2-1)\sqrt{\frac{1-\omega_1^2}{\omega_2}} & i(1-\omega_1)\sqrt{\frac{\omega_2^2-1}{\omega_1}} \\ -(1-\omega_1)\sqrt{\frac{\omega_2^2-1}{\omega_1}} & (\omega_2-1)\sqrt{\frac{1-\omega_1^2}{\omega_2}} \end{pmatrix} \tag{97}$$

Having diagonalized the Hamiltonian, we can write down the energy eigenstates:

$$|n,m\rangle = \frac{(X^\dagger)^n(Y^\dagger)^m}{\sqrt{n!m!}}|0\rangle, \qquad H|n,m\rangle = [\omega_1(n+1/2) + \omega_2(m+1/2)]|n,m\rangle \tag{98}$$

A few comments are in order. First, although the oscillator frequencies ω_1 and ω_2 look different from the frequencies ω_u and ω_v calculated in the previous section, they are in fact the same, as they should be. More precisely, if $\omega_z^2 < \omega_0^2 + \omega_x^2$, then $\omega_1 = \omega_u$ and $\omega_2 = \omega_v$; if $\omega_z^2 > \omega_0^2 + \omega_x^2$, then $\omega_1 = \omega_v$ and $\omega_2 = \omega_u$. This branching behavior can be traced to the definition of α which is singular at $\omega_z^2 = \omega_0^2 + \omega_x^2$. Returning to the comment made towards the end of Section 6.1.1, we now recognize that in the limit $\omega_x \to 0$, if $\omega_0 > \omega_z$ then $(X^\dagger, X)$ can be identified as the raising/lowering operators in the plane and $(Y^\dagger, Y)$ can be identified as the raising/lowering operators along the z-axis. If $\omega_0 < \omega_z$, the roles of X and Y are reversed.

Second, the energy eigenstates that we just introduced are labeled with two quantum numbers. A three-dimensional quantum problem generally has three quantum numbers, which we verified explicitly in the previous section when we found that the energy eigenstates depended on q, the canonical momentum in the y direction. We therefore expect a third set of raising/lowering operators. Indeed, we find that

$$\begin{aligned} c &= a^\dagger - i\sqrt{\frac{B_z}{2}}\left(x + iy - \frac{B_x}{B_z}z\right) \\ &= \alpha^\dagger - i\sqrt{\frac{B_z}{2}}\left(x + iy\right) \end{aligned} \tag{99}$$

satisfies $[c, c^\dagger] = 1$ and $[c, a] = [c, a^\dagger] = [c, b] = [c, b^\dagger] = 0$. Since X and Y are linear combinations of a, $a^\dagger$, b and $b^\dagger$ only, we see that $(X^\dagger, X)$, $(Y^\dagger, Y)$ and $(c^\dagger, c)$ define three independent raising/lowering operators. By acting with $c^\dagger$ repeatedly on the $|n, m\rangle$ eigenstates defined above, we thus get the desired (nearly) infinite tower of states with identical energy.

Finally, when computing the response functions and stress tensor, it is helpful to have the momenta expressed in terms of the ladder operators. Using the definitions of a, b and α, we find

$$\pi_x = \sqrt{\frac{B_z}{2}}(a + a^\dagger) = \sqrt{\frac{B_z}{2}}(\alpha + \alpha^\dagger) \tag{100}$$

$$\pi_y = i\sqrt{\frac{B_z}{2}}(a - a^\dagger) = i\sqrt{\frac{B_z}{2}}\left((\alpha - \alpha^\dagger) + \frac{\omega_x}{\sqrt{\omega_z\omega_0}}(b^\dagger - b)\right) \tag{101}$$

$$\pi_z = \sqrt{\frac{m\omega_0}{2}}(b + b^\dagger) \tag{102}$$

Using the expressions for the components of U, we can express the momenta as linear combinations of the $(X^\dagger, X)$ and $(Y^\dagger, Y)$ ladder operators.

6.2 Hall conductivity

We have developed all the tools necessary to start computing response functions. Having seen the nature of the computations in Section 5, we'll leave out more of the details in the present section, especially since the tilted field calculations are more tedious.

Using the definition of the Hall conductivity given in equation (51), and the expressions for the momenta in terms of the ladder operators given in equations (100)-(103), we find that Hall conductivity simplifies dramatically:

$$\sigma^H_{xz} = \sigma^H_{zy} = 0, \qquad \sigma^H_{xy} = \frac{e\bar{n}}{B_z} \tag{103}$$

This is an exact result. As we can see, the Hall conductivity does not depend on the tilted component of the magnetic field B_x nor on the confining potential ω_0. Thus, the Hall conductivity in the ground state does not respond to the anisotropy introduced by increasing the magnetic field component in the plane. This mimics the result for the band mass anisotropy.

It is somewhat surprising that the Hall conductivity does not depend on the confining potential strength ω_0. If there were no confining potential ($\omega_0 = 0$), then we know that the Hall conductivity would be that of the isotropic case properly rotated. Namely,

$$\begin{aligned}\sigma^H_{\mu\nu} &= \frac{e\bar{n}}{B}\begin{pmatrix} \cos\alpha & 0 & -\sin\alpha \\ 0 & 1 & 0 \\ \sin\alpha & 0 & \cos\alpha \end{pmatrix}\begin{pmatrix} 0 & 1 & 0 \\ -1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}\begin{pmatrix} \cos\alpha & 0 & \sin\alpha \\ 0 & 1 & 0 \\ -\sin\alpha & 0 & \cos\alpha \end{pmatrix} \\ &= \frac{e\bar{n}}{B}\begin{pmatrix} 0 & \cos\alpha & 0 \\ -\cos\alpha & 0 & -\sin\alpha \\ 0 & \sin\alpha & 0 \end{pmatrix}\end{aligned} \tag{104}$$

where $B = \sqrt{B_z^2 + B_x^2}$ and $\alpha = \arctan(B_x/B_z)$. Thus, we find

$$\sigma^H_{\mu\nu} = \frac{e\bar{n}}{B_z^2 + B_x^2}\begin{pmatrix} 0 & B_z & 0 \\ -B_z & 0 & -B_x \\ 0 & B_x & 0 \end{pmatrix} \tag{105}$$

Clearly, the Hall conductivity tensor is discontinuous at $\omega_0 = 0$ except when $B_x = 0$.

We can present a qualitative argument why σ^H_{xz} and σ^H_{yz} should be 0. These components of the conductivity tensor capture how the current in the z-direction responds to an applied electric field in the x or y direction. In order for a state to carry current in the z direction, it must be extended (e.g., like a plane wave) rather than localized (e.g., like a Gaussian packet) in the z-direction. However, it is not possible for a state with finite energy to be extended in a quadratic well. Consider the simple case of the one-dimensional quantum harmonic oscillator. The average energy of a state $|\psi\rangle$ is then $\langle E\rangle \sim \langle x^2\rangle/2 + \langle p^2\rangle/2$. Since $\langle p^2\rangle$ is strictly positive and $\Delta^2 = \langle x^2\rangle - \langle x\rangle^2$ is the variance of the position of the particle, we see that if the state is extended–so that the variance is infinite–then the average energy is also infinite.

The Hall conductivity result also reflects the fact that the energy gap above the ground state closes as $\omega_0 \to 0$. This is because the number of states available to the ground state electrons is fixed while the gap is open. When the gap closes, the number of available increases drastically and only then is the conductivity able to change. We can explicitly check that $\omega_v \to 0$ as $\omega_0 \to 0$.

6.3 Stress-energy tensor

We next derive the form of the stress-energy tensor for the tilted field Hamiltonian. We proceed in the same manner as for the band mass anisotropic Hamiltonian. Defining the momentum density operator for a single particle to be $g_\mu(\vec{r}) = \frac{1}{2}\{\pi_\mu, \delta(\vec{r} - \vec{x})\}$, we have

$$\frac{\partial g_\mu}{\partial t} = i[H, g_\mu] = \frac{i}{4m}\left([\pi_\nu\pi_\nu, \{\pi_\mu, \delta(\vec{r} - \vec{x})\}] + m^2\omega_0^2[z^2, \{\pi_\mu, \delta(\vec{r} - \vec{x})\}]\right) \tag{106}$$

Noting $[\pi_\mu, \pi_\nu] = -i\epsilon_{\mu\nu\rho}B_\rho$, the first term reduces to

$$\begin{aligned}[\pi_\nu\pi_\nu, \{\pi_\mu, \delta(\vec{r} - \vec{x})\}] &= \{[\pi_\nu\pi_\nu, \delta(\vec{r} - \vec{x})], \pi_\mu\} + \{[\pi_\nu\pi_\nu, \pi_\mu], \delta(\vec{r} - \vec{x})\} \\ &= -i\{\{\pi_\nu, \frac{\partial}{\partial x_\nu}\delta(\vec{r} - \vec{x})\}, \pi_\mu\} + 2i\epsilon_{\nu\mu\rho}B_\rho\{\pi_\nu, \delta(\vec{r} - \vec{x})\} \\ &= i\frac{\partial}{\partial r_\nu}\{\{\pi_\nu, \delta(\vec{r} - \vec{x})\}, \pi_\mu\} + 4i\epsilon_{\mu\nu\rho}g_\nu B_\rho\end{aligned} \tag{107}$$

Meanwhile, the second term is

$$[z^2, \{\pi_\mu, \delta(\vec{r} - \vec{x})\}] = \{[z^2, \pi_\mu], \delta(\vec{r} - \vec{x})\} = 4i\delta_{\mu 3}z\delta(\vec{r} - \vec{x}) \tag{108}$$

Therefore,

$$\frac{\partial g_\mu}{\partial t} = -\frac{\partial}{\partial r_\nu}\left(\frac{1}{4m}\{\{\pi_\nu, \delta(\vec{r}-\vec{x})\}, \pi_\mu\}\right) - \frac{1}{m}\epsilon_{\mu\nu\rho} g_\nu B_\rho - m\omega_0^2 z \delta_{\mu 3} \tag{109}$$

The second term is the Lorentz force and the third term is the confining force, so we see that the first term is the stress tensor:

$$\tau_{\mu\nu} = \frac{1}{4m}\{\{\pi_\mu, \delta(\vec{r}-\vec{x})\}, \pi_\nu\} \tag{110}$$

Integrating over space to get $T_{\mu\nu}$, we find

$$T_{\mu\nu} = \int d^3r \tau_{\mu\nu} = \frac{1}{2m}\sum_{i=1}^{N}\{\pi_\mu^i, \pi_\nu^i\} \tag{111}$$

where in the last step we've included the contributions of all N electrons. The result is the same as for the isotropic case.

We can now determine the stress tensor components in the ground state. The components that interest us the most are the xx, xy and yy components, which capture the physics in the plane. To leading order in k and l, they are given by

$$\langle\tau_{xx}\rangle_0 = \frac{\hbar\omega_z\bar{n}}{2}\left(1 - \frac{k^2l^2}{2}\right) \tag{112}$$

$$\langle\tau_{xy}\rangle_0 = 0 \tag{113}$$

$$\langle\tau_{yy}\rangle_0 = \frac{\hbar\omega_z\bar{n}}{2}\left(1 + k^2 l - \frac{3}{2}k^2l^2\right) \tag{114}$$

We have reinserted factors of $\hbar$, included the contributions of all N electrons and divided by the volume to get the intensive rather than the integrated stress tensor. Let us analyze the results. First, we notice that the components reduce to what we have for the band mass/isotropic cases when $k \to 0$ or $l \to 0$, as they intuitively should. However, as long as there is a slight tilt and a finite confining potential strength (i.e., $k \neq 0$, $l \neq 0$), the tilted field stress tensor is not that of a homogeneous fluid, since it is clearly not proportional to the identity tensor. In particular, there is no effective mass tensor whose resulting stress tensor behaves like the tilted field stress tensor.

We also notice that the the sub-leading order term in $\langle\tau_{xx}\rangle_0$ goes like k^2l^2 while the sub-leading order term in $\langle\tau_{yy}\rangle_0$ goes like k^2l. If we examine the derivation carefully, we see that the k^2l term in the $\langle\tau_{yy}\rangle_0$ component comes from the raising/lowering operators along the z-axis (i.e., $(Y^\dagger, Y)$ for the case $\omega_0 > \omega_z$). Thus, the k^2l term can be thought of as a residual of the fact that we working with a 2D projection of a 3D system rather than a bona fide 2D system. We will encounter many other such residuals.

It is also useful to calculate the projected form of the stress tensor operators. To do so, we imagine that we are working in the planar subspace of the full particle Hilbert space. Since we are interested in the regime $l \ll 1$, which means $\omega_z \ll \omega_0$, the raising/lowering operators in the plane are $(X^\dagger, X)$ and the raising/lowering operators along the z-axis are $(Y^\dagger, Y)$. Therefore, to project into the plane, we restrict ourselves to the subspace with basis states given by $|n, m\rangle \sim (X^\dagger)^n (c^\dagger)^m |0\rangle$. In this Hilbert space, any expression normal ordered[15] in $Y^\dagger$ and Y is 0. Thus, to extract the

[15] Normal ordering means we use the ladder commutation relations to write all the annihilation operators to the right of all the creation operators. Of the resulting sum, only the c-number terms contribute to the ground-state matrix elements. For instance $YYY^\dagger = YY^\dagger Y + Y[Y, Y^\dagger] = Y^\dagger YY + [Y, Y^\dagger]Y + Y = Y^\dagger YY + 2Y$. The final expression is normal ordered, and evaluates to 0 for any ground state matrix element.

projected stress tensor operators we normal order the Y ladder operators and afterwards drop all Y-terms. Finally, we expand the coefficients to the leading non-trivial order. We get:

$$\tau_{xx} = -\frac{\hbar\omega_z\bar{n}}{2}\left(1 - \frac{k^2l^2}{2}\right)\left(X^\dagger - X\right)^2 \tag{115}$$

$$\tau_{xy} = \frac{i\hbar\omega_z\bar{n}}{2}\left(1 - k^2l^2\right)\left[(X^\dagger)^2 - X^2\right] \tag{116}$$

$$\tau_{yy} = \frac{\hbar\omega_z\bar{n}}{2}\left(\left(1 - \frac{3}{2}k^2l^2\right)(X^\dagger + X)^2 + \boxed{k^2l}\right) \tag{117}$$

The boxed term comes from a $YY^\dagger$ term, and therefore captures the fact that we are projecting a 3D system into 2D.[16]

Although our projection procedure may provide insights, one must be careful not to assume it is a more powerful tool than it actually is. For instance, the commutator of projected operators is not equal to the projection of the commutators, because the commutator of two normal-ordered expressions is not necessarily normal ordered (e.g., $[Y, Y^\dagger] = YY^\dagger - Y^\dagger Y$). Thus, when we evaluate the Hall viscosity in the next section, we would risk missing terms if we would use the projected forms of the stress tensor operators.

6.4 Hall viscosity

We can calculate the Hall viscosity using either the gauge-dependent wave functions or the ladder operators. We arrive at the same answers either way. We'll spare the details and state the results. We find that the non-zero components of the (intensive) Hall viscosity are given by

$$\eta^H_{1112} = -\frac{\bar{n}}{4}\left(1 - \frac{1}{2}k^2l^2\right) \tag{118}$$

$$\eta^H_{1222} = -\frac{\bar{n}}{4}\left(1 - \frac{3}{2}k^2l^2\right) \tag{119}$$

The first check is that for $k \to 0$ or $l \to 0$, the results reduce to that of the isotropic 2D system, as expected. We can also make a quaint observation: the perpendicular component of the magnetic field determines the magnitude of the Hall viscosity (i.e., $\bar{n}$ depends only on ω_z), while the strength of the parallel component of the magnetic field relative to the confining potential determines the deviation of the Hall viscosity from the isotropic form (i.e., $k^2l^2 = \frac{\omega_x^2}{\omega_0^2}$ is independent of ω_z).

It is also straightforward to calculate the reduced Hall viscosity. We find:

$$\eta^H_{ab} = \begin{pmatrix} \eta^H_{1222} & 0 \\ 0 & \eta^H_{1112} \end{pmatrix} = -\frac{\bar{n}}{4}\begin{pmatrix} 1 - \frac{3}{2}k^2l^2 & 0 \\ 0 & 1 - \frac{1}{2}k^2l^2 \end{pmatrix} \tag{120}$$

We will use this result in Section 7.3 to extract an effective mass tensor for the tilted field system.

Finally, we return to a subtlety we have delayed: the contact term in the Hall viscosity. We found in equation (50) that the Hall viscosity has a component given by $\frac{i}{\omega^+}\left(\delta_{\nu\rho}\left\langle T_{\mu\sigma}\right\rangle_0 - \delta_{\mu\sigma}\left\langle T_{\rho\nu}\right\rangle_0\right)$. Unlike for the band mass system, it is non-trivial for the tilted field system. Specifically, we find one non-zero term:

$$\eta^S_{1221}(\omega) = \frac{i}{\omega^+}\left(\left\langle\tau_{xx}\right\rangle_0 - \left\langle\tau_{yy}\right\rangle_0\right) = -\frac{i\hbar\omega_z\bar{n}}{2\omega^+}k^2l(1 - l) \tag{121}$$

[16] We also determined the projected forms of the τ_{xz}, τ_{yz} and τ_{zz} operators. However, they don't play a major role in our narrative, so they are relegated to Appendix B, which includes other results that we calculated but will not use.

Notice the $\frac{1}{\omega}$ divergence as $\omega \to 0$, which means the contact term should be interpreted as a Hall shear modulus rather than a Hall viscosity. Namely, from $\eta^S(\omega)\frac{\partial\lambda}{\partial t}(\omega) = (-i\omega\eta^S(\omega))\lambda(\omega)$, we see that $-i\omega\eta^S$ is a well-behaved linear response function that captures the behavior of the stress tensor under a static rather than a dynamic strain. This Hall shear modulus (if correct) represents a real physical effect that is rather unusual. It does not have an analog in homogeneous fluids, like the 2DEG in the band mass system. However, it is consistent with our results for the components of the ground state stress tensor, given in equations (112)-(114); since $\langle\tau_{xx}\rangle_0$ and $\langle\tau_{yy}\rangle_0$ are not identical, a static rotation that changes $x \to y$ and $y \to -x$ changes the stress tensor. We believe the Hall shear may be experimentally measurable.

7 Is there an effective band mass tensor for the tilted field?

In this, the last major section of the paper, the work we carried out previously really begins to pay off. Although our analyses of the Hall conductivity and Hall viscosity in the band mass and tilted field systems certainly demonstrate the versatility of the linear response formalism developed in Sections 3 and 4, the two systems have a more salient connection than merely being two alternative model systems. In particular, answering the question of whether the two systems can be related– whether there is a mapping from tilted field systems to band mass systems that behave almost identically– may pay high scientific dividends. For instance, while theoretical quantum Hall research into anisotropy frequently works with the band mass system because the mathematical framework is particularly simple, the simplest way for experimentalists to introduce anisotropy is to tilt their samples relative to their electromagnets. Thus, a mapping from tilted field systems to band mass systems would help bridge experiment and theory. Furthermore, from a purely theoretical viewpoint, whether a mapping exists may provide insights into general attempts to project 3D systems into simpler 2D systems.

We have certainly seen promising similarities between the band mass and tilted field systems. They both reduce to the isotropic system as their respective anisotropies are lifted, the ground state for both is protected by an energy gap, and in both cases the Hall conductivity is unaffected by the anisotropy. However, we have also seen indications of the two systems having certain fundamentally different properties, like the different behaviors of the ground state stress tensors.

To compare the two anisotropies systematically, we proceed by extracting an effective anisotropy tensor (i.e., an effective $\tilde{m}_{ab}$ in the language of Section 5) from the tilted field system in three different ways: by looking at the "lowest angular momentum" ground state wavefunction, by looking at the stress tensor operators, and by looking at the Hall viscosity results. For the third step, we will have to briefly introduce work done by A. Gromov, S. Geraedts and B. Bradlyn in [6]. What we find is that all three methods give different anisotropy matrices, further hinting at the fundamental 3-dimensionality of the tilted field system. We argue that if a single anisotropy matrix could effectively represent the tilted field system, it would probably be the one extracted from the Hall viscosity since it is the most physical.

7.1 Effective anisotropy from wavefunction

In our first method of extracting an anisotropy matrix from the tilted field system, we compare the form of the wavefunction of the band mass state annihilated by the a and b lowering operators introduced in Section 5.1 to the form of the wavefunction of the tilted field state annihilated by the X, Y and c lowering operators introduced in Section 6.1.2. We refer to these states as the "lowest angular momentum" ground states because in the isotropic case the lowering operator acting within an energy eigenspace (which corresponds to a and c) lowers the angular momentum of a state by $\hbar$

(see Section 1.4.3. of [18]). Therefore, the state it annihilates has the lowest angular momentum. Unfortunately, we use a and b to refer to ladder operators in both the band mass and tilted field systems. One must be careful not to confuse them below.

First, we need to solve for the state $|\psi\rangle$ in the band mass system satisfying $a|\psi\rangle = b|\psi\rangle = 0$. Let us work in position space in the frame where the mass tensor is diagonal: $\tilde{m}_{ab} = \frac{1}{m}\text{diag}\left(\frac{1}{\alpha}, \alpha\right)$. Let us also use the symmetric gauge: $\vec{A} = B(-y/2, x/2)$. We find that the wavefunction satisfies the differential equations

$$0 = \left[\frac{i}{\sqrt{\alpha}}\left(\frac{1}{i}\frac{\partial}{\partial x} - \frac{By}{2}\right) + \sqrt{\alpha}\left(\frac{1}{i}\frac{\partial}{\partial y} + \frac{Bx}{2}\right)\right]\psi(x,y) \tag{122}$$

$$0 = \left[\frac{i}{\sqrt{\alpha}}\left(\frac{1}{i}\frac{\partial}{\partial x} + \frac{By}{2}\right) - \sqrt{\alpha}\left(\frac{1}{i}\frac{\partial}{\partial y} - \frac{Bx}{2}\right)\right]\psi(x,y) \tag{123}$$

Introducing the re-scaled variables $\tilde{x} = \sqrt{\alpha}x$, $\tilde{y} = \frac{y}{\sqrt{\alpha}}$ and the complex variable $w = \tilde{x} + i\tilde{y}$, in which case $\frac{\partial}{\partial w} = \frac{1}{2}\left(\frac{\partial}{\partial x} - i\frac{\partial}{\partial y}\right)$, we see that the equations reduce to $0 = \left[2\frac{\partial}{\partial w} + \frac{Bw^*}{2}\right]\psi(w, w^*) = \left[2\frac{\partial}{\partial w^*} + \frac{Bw}{2}\right]\psi(w, w^*)$. Up to a normalization, the lowest angular momentum ground state is then

$$\psi(x,y) = \exp\left(-\frac{B|w|^2}{4}\right) = \exp\left(-\frac{B(\tilde{x}^2 + \tilde{y}^2)}{4}\right) = \exp\left(-\frac{B}{4}(\alpha x^2 + \frac{y^2}{\alpha})\right) \tag{124}$$

Second, we need to solve for the state $|\psi\rangle$ in the tilted field system satisfying $X|\psi\rangle = Y|\psi\rangle = c|\psi\rangle = 0$. Let us work in position space and in the gauge $\vec{A} = (-B_z y/2, B_z x/2 - B_x z, 0)$. Again introducing the complex variable $w = x + iy$, we find that the α, b and c operators that we used in Section 6.1.2 take the form

$$\alpha = \frac{1}{i\sqrt{2B_z}}\left(2\frac{\partial}{\partial w} + \frac{B_z}{2}w^*\right), \quad b = \frac{1}{i\sqrt{2m\omega_0}}\left(\frac{\partial}{\partial z} + m\omega_0 z\right), \quad c = \frac{1}{i\sqrt{2B_z}}\left(2\frac{\partial}{\partial w^*} + \frac{B_z}{2}w\right) \tag{125}$$

Since equations (90) and (94) – (97) let us write X and Y as linear combinations of α, $\alpha^\dagger$, b and $b^\dagger$, it becomes straightforward to solve for $\psi(x,y,z)$. We find that up to normalization the lowest angular momentum ground state is

$$\psi(x,y,z) = \exp\left(-\frac{B_z}{4}|w|^2 - C_1 m\omega_0 z^2 + C_2 B_z w^2 + C_3\sqrt{m\omega_0 B_z}wz\right) \tag{126}$$

where to leading order $C_1 = 1 + k^2l^2$, $C_2 = \frac{k^2l^2}{16}$ and $C_3 = \frac{1}{2}kl^{3/2}$. To project into the x-y plane, we set $z = 0$.[17] Furthermore, we can use the gauge freedom to get rid of any imaginary components in the exponential. We find that the projected wave function is

$$\psi_P(x,y) = \exp\left(-\frac{B_z}{4}\left[\left(1 - \frac{k^2l^2}{4}\right)x^2 + \left(1 + \frac{k^2l^2}{4}\right)y^2\right]\right) \tag{127}$$

Finally, comparing equations (124) and (127), we can set $\alpha = 1 - \frac{k^2l^2}{4}$. We thus identify our first anisotropy matrix:

$$v_{ab} = \begin{pmatrix} 1 + \frac{k^2l^2}{4} & 0 \\ 0 & 1 - \frac{k^2l^2}{4} \end{pmatrix} \tag{128}$$

where we introduce the notation $v_{ab} \equiv m\tilde{m}_{ab}$, the effective band mass tensor rescaled to have unit determinant.

[17] An alternative method of projection is to integrate over z, which is particularly simple because the ground state wavefunction is Gaussian in z. We find that the two methods give the same leading order result.

7.2 Effective anisotropy from stress tensor

We can extract anisotropy from the form of the projected stress tensor operators using a particularly simple approach. We take inspiration from the isotropic system, and define new physical momenta: $\Pi_x = i\sqrt{\frac{B_z}{2}}(X^\dagger - X)$ and $\Pi_y = \sqrt{\frac{B_z}{2}}(X^\dagger + X)$. Looking at equations (115) – (117), we see that the projected stress tensors become

$$\tau_{xx} \sim \left(1 - \frac{k^2 l^2}{2}\right)\Pi_x^2, \qquad \tau_{yy} \sim \left(1 - \frac{3}{2}k^2 l^2\right)\Pi_y^2 + 2\frac{\omega_x^2}{\omega_0\omega_z}, \qquad \tau_{xy} \sim \left(1 - k^2 l^2\right)\frac{1}{2}\{\Pi_x, \Pi_y\} \tag{129}$$

Recalling that $\tau_{ab} \sim \tilde{m}_{bc}\{\pi_a, \pi_c\}$ for the band mass system, and examining the (momentum dependent parts of the) xx and yy components of the tilted field stress tensor, we deduce that the effective band mass tensor is

$$\tilde{m}_{ab} = \frac{1}{m}\begin{pmatrix} 1 - \frac{k^2l^2}{2} & 0 \\ 0 & 1 - \frac{3}{2}k^2l^2 \end{pmatrix} = \frac{1}{m}(1 - k^2l^2)\begin{pmatrix} 1 + \frac{k^2l^2}{2} & 0 \\ 0 & 1 - \frac{k^2l^2}{2} \end{pmatrix} \tag{130}$$

which yields an effective mass $m^* = m(1 + k^2l^2)$ and the unit determinant anisotropy matrix

$$v_{ab} = \begin{pmatrix} 1 + \frac{k^2l^2}{2} & 0 \\ 0 & 1 - \frac{k^2l^2}{2} \end{pmatrix} \tag{131}$$

Notice that the tilted field τ_{xy} is not given correctly by $v_{yy}\{\Pi_x, \Pi_y\}$ nor by $v_{xx}\{\Pi_y, \Pi_x\}$, but rather by the average: $\tau_{xy} = \frac{1}{2}(v_{xx} + v_{yy})\{\Pi_x, \Pi_y\}$.

Despite the need to fudge the correspondence between the tilted field and the band mass system, the extracted anisotropy matrix might still be useful. A bigger problem with the correspondence is the fact that the constant $\frac{2\omega_x^2}{\omega_0\omega_z}$ term in the τ_{yy} component has no analog in the band mass system. We interpret it as an intrinsically 3D effect that cannot be reproduced in two dimensions, thus reflecting a limitation of the projection procedure.

Incidentally, since we defined the stress tensors via a continuity equation, the stress tensor is defined only up to a divergenceless term. In three dimensions, for instance, we may use the freedom to transform a given stress tensor via $\tau_{\mu\nu} \to \tau_{\mu\nu} + \epsilon_{\mu\nu\rho}\partial_\rho\phi$ for any scalar function ϕ because $\partial_\mu(\epsilon_{\mu\nu\rho}\partial_\rho\phi) = 0$. One may hope that one can use this freedom to account for the τ_{xy} and constant term discrepancies between the tilted field and band mass. We are skeptical, however. In particular, if we make the seemingly reasonable assumption that the stress tensor has a well-defined behavior at infinity (which means that the auxiliary field ϕ that we introduced vanishes at infinity), then the *integrated* stress tensor is not affected by adding divergenceless terms to the intensive stress tensor and the integrated stress tensor also features the discrepancies.

7.3 Effective anisotropy from Hall viscosity

Our final method of extracting an anisotropy matrix for the tilted field system requires some exposition. In [6], Gromov, Geraedts and Bradlyn develop a general formalism for understanding the role of anisotropy in quantum Hall states. They consider a low energy effective field theory in two dimensions with two metrics, the usual geometrical metric and a second metric characterizing the anisotropy. They assume that the anisotropy metric is a symmetric two-component tensor with unit determinant, but impose no restrictions on where it originates from (e.g., from a mass tensor, from anisotropic interactions, or from a tilt in the magnetic field). The result that is most relevant to our discussion is the following expression for the contracted Hall viscosity tensor:

$$\eta^H_{ab} = -\frac{\bar{n}}{2}\left[s\delta_{ab} + \varsigma v_{ab} + \xi(v_{ac}v_{cb} - \delta_{ab})\right] \tag{132}$$

The coefficients s, ς and ξ are coupling constants that capture how the two metrics (in particular, how the two spin and Christoffel connections) enter into the low energy effective action describing the quantum Hall state. Importantly, s and ς are quantized and sum to one-half of the topological invariant known as the shift, $\mathcal{S}$. In the case of the ground state, $s + \varsigma = \frac{\mathcal{S}}{2} = \frac{1}{2}$. Meanwhile, v_{ab} is the unit determinant and symmetric anisotropy tensor we want to solve for using our Hall viscosity results.

In equation (120), we give the components of the contracted Hall tensor for the tilted field system after projecting into the plane. Counting the number of unknowns ($3 + 2 = 5$ for the three coefficients and two independent components of a unit determinant symmetric 2×2 matrix) and the number of knowns ($3 + 1 = 4$ for the three independent components of the contracted Hall tensor and the relation between s and ς), the system of equations seems underdetermined. Therefore, we make an assumption motivated by the observation in [6] that s and ς correspond to anisotropy introduced via the interactions and via the band mass tensor, respectively. Since we seek an effective band mass tensor for the tilted field, we set $s = 0$, in which case $\varsigma = \frac{1}{2}$.

It is not too difficult to show that v_{ab} must be diagonal given our contracted Hall tensor, so we write $v_{ab} = \mathrm{diag}\,(\alpha, 1/\alpha)$. Substituting into equation (132) and using equation (120), we find two possible solutions. To leading order, $\alpha = 1 \pm \sqrt{2}kl$ and $\xi = -\frac{1}{4}(1 \pm k^2l^2/2/\sqrt{2})$. Thus, the anisotropy matrix is given by

$$v_{ab} = \begin{pmatrix} 1 \pm \sqrt{2}kl & 0 \\ 0 & 1 \mp \sqrt{2}kl \end{pmatrix} \tag{133}$$

The most striking feature of the anisotropy tensor is that, unlike the other two extracted anisotropy matrices, it depends on $kl = \frac{\omega_x}{\omega_0}$ rather than k^2l^2. Thus, the anisotropy depends on the direction of the parallel component of the magnetic field as well as on its magnitude. This suggests that a full theory of the anisotropy of the projected tilted field system should look at vector anisotropy rather than quadropolar anisotropy, which is what the decomposition in equation (132) is based on.

Although it is not our primary focus, we can also comment on the result that $\xi \to -\frac{1}{4}$ as the anisotropy is lifted, which is interesting given the discussion in [6]. In the paper, the authors apply their formalism to the band mass system and various fractional quantum Hall states, and for each they find $\xi \approx 0$ (although for some of the fractional quantum Hall states, it seems that the deviation of ξ from 0, though small, is statistically significant). They mention that one cannot use effective field theoretic arguments to conclude that ξ must be 0, and we have found a simple system that validates that claim.

7.4 Probably not

We have identified three matrices that capture the anisotropy of the tilted field when it is treated as a 2D system. They are given in equations (128), (131) and (133), and the key observation is that they are all different. This suggests that the answer to the question of whether there exists an effective band mass for the tilted field is no. Or, at least, there is no single consistent effective band mass that properly captures the behavior of the wavefunction, stress tensor, and Hall viscosity. If a single mass tensor would have to be identified, we think it would be most logical to use the one extracted using the bi-metric formalism, since it characterizes a readily observable property.

The properties of a particular ground state wavefunction and the stress tensor are more difficult to measure than the Hall viscosity.

Coupled with the two unusual physical effects that we identified for the tilted field system– the constant-like term in the stress tensor and the Hall shear modulus– the inability to identify a unique band mass tensor for the tilted field indicates that by attempting to treat the tilted field system as a 2D system, we lose potentially significant physics.

Before we conclude our work, it should be emphasized that the question of band mass/tilted field correspondence should be investigated further. For instance, other papers [13], [21] cite an effective band mass tensor that is identical (up to a normalization to make the matrix have unit determinant) to the one we determined from the stress tensor. We suspect those papers either looked at the wavefunction or the Coulomb interaction to identify the anisotropy matrix. If they did the former, their results disagree with ours (see equation (128)), which may reflect a minor error in Section 7.1. Furthermore, if we compare the contracted Hall tensors for the band mass and tilted field systems directly (see equations (76) and (120)), we can try to extract a fifth anisotropy matrix. Up to a normalization (of $1 - k^2 l^2$) to make the matrix have unit determinant, we find the same result as using the stress tensor. We did not discuss this method on par with the others, because the normalization must be interpreted as shifting the filling density $\bar{n}$, which is undesirable given the need for the tilted field system and its effective band mass counterpart to have the same Hall conductivity and stress tensor in the isotropic limit.

Nonetheless, it seems possible that, with the potential correction of a few errors and the adoption of a slightly different perspective, one could identify a pretty universal anisotropy matrix for the tilted field. The big question then would be, how far can the consequences of that effective band mass tensor be pushed (which we've already started answering) and why does the bi-metric formalism give a different result? If, however, the multitude of effective mass tensors are robust to further probing, they might be interpreted as capturing different physical effects, in much the same way that in solid state physics more generally one can identify multiple effective masses for a particle, like the Drude effective mass, the cyclotron effective mass, the specific heat effective mass, etc, each corresponding to a different property of the system. Properly understanding the different effective mass tensors should therefore lead to further insight into the tilted field system.

8 Conclusions

In this paper we used linear response theory to determine the Hall conductivity and Hall viscosity for two simple quantum Hall systems with anisotropy. We used our results to attempt to identify a correspondence between the two systems. It seems no simple consistent correspondence exists. Along with various unusual and potentially experimentally verifiable effects that we have identified, like the Hall shear modulus in particular, the lack of a correspondence suggests that the tilted system behaves in certain fundamentally different ways from the band mass system. It seems promising to continue to investigate the two systems, and to try to find a cohesive theoretical framework for understanding the different results we have identified.

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Appendix A: Fourier transform of spatial convolution

For simplicity, let us work in one dimension. Assume $h(x) = \int dx' f(x,x')g(x')$ where $f(x,x')$ is translationally invariant. That is, $f(x+\epsilon, x'+\epsilon) = f(x,x')$ for any ϵ. Writing $f(x,x')$ in terms of its Fourier transform, we have

$$f(x,x') = \int_{-\infty}^{\infty} \frac{dq dq'}{(2\pi)^2} e^{-iqx} e^{-iq'x'} f(q,q') \tag{134}$$

Translational invariance implies $f(q,q') = 0$ whenever $e^{-i(q+q')\epsilon} \neq 1$ for all ϵ, which is true whenever $q' \neq -q$. Thus, we conclude that $f(q,q')$ is proportional to $\delta(q+q')$. Let us write $f(q,q') = 2\pi \tilde{f}(q)\delta(q+q')$. Thus, $f(x,x')$ evaluates to

$$f(x,x') = \int_{-\infty}^{\infty} \frac{dq}{2\pi} e^{-iq(x-x')} \tilde{f}(q) \tag{135}$$

Taking the Fourier transform of $h(x,x')$, we have

$$h(q) = \int \frac{dx dx' dq'}{2\pi} e^{-iq'(x-x')} e^{iqx} \tilde{f}(q') g(x') = \tilde{f}(q) \int dx' e^{iqx'} g(x') = \tilde{f}(q) g(q)$$

Multiplying both sides by $2\pi\delta(q+q')$, we thus get $2\pi\delta(q+q')h(q) = f(q,q')g(q)$. Letting $q' = -q$, we find $2\pi\delta(0)h(q) = f(q,-q)g(q)$. For a finite system with volume V, we can interpret the delta function evaluated at 0 to be related to the volume. Namely, $2\pi\delta(q=0) = \int d^d x e^{ix(q=0)} = V$. Finally, the result we wanted to show:

$$h(q) = \frac{1}{V} f(q,-q) g(q) \tag{136}$$

Appendix B: More tilted field results

In this section we list some results we calculated for the tilted field, but did not have time to interpret meaningfully. First, the projected forms of τ_{zz}, τ_{xz} and τ_{yz} components of the stress tensor are

$$\tau_{zz} = \frac{\hbar\omega_0 \bar{n}}{2} \left(k^2 l^5 (iX^\dagger - iX)^2 + \boxed{\left(1 + \frac{k^2 l^2}{2}\right)} \right) \tag{137}$$

$$\tau_{xz} = \frac{\hbar\omega_0 \bar{n}}{2} k l^2 \left(l \left(1 + l^2\right) (iX - iX^\dagger)^2 - \boxed{\left(1 + l^2\right)} \right) \tag{138}$$

$$\tau_{yz} = i \frac{\hbar\omega_0 \bar{n}}{2} k l^3 ((X^\dagger)^2 - X^2) \tag{139}$$

where, again, the terms that are boxed originated from Y terms in the calculation, and thus are residuals from the 3D to 2D projection process.

Second, the non-zero components of the Hall viscosity with at least one component out of the plane, expanded to leading order in k and l, are given by

$$\eta_{2223} \sim \frac{3}{4} k l^2, \qquad \eta_{1213} \sim \frac{1}{4} k l^2, \qquad \eta_{1233} \sim -\frac{1}{4} k^2 l^3 \tag{140}$$

$$\eta_{1123} \sim \frac{1}{4} k l^2, \qquad \eta_{1323} \sim \frac{1}{2} l, \qquad \eta_{2333} \sim \frac{1}{4} k l \tag{141}$$

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