Section A. Mechanics

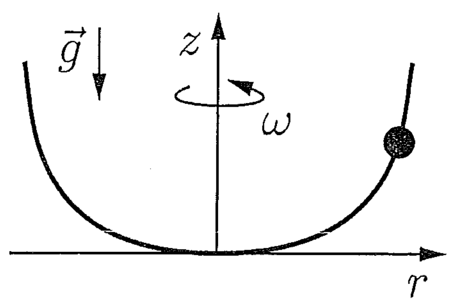
1. Bead sliding on a rotating wire

A bead with mass m slides without friction on a "U-shaped" wire which lies in a plane parallel to the z-axis, and follows a path

$$z = \frac{r^4}{a^3}$$

where r is the distance from the z-axis to a point on the wire at height z. The downwards acceleration g due to gravity is parallel to the z-axis.

The wire is now constrained to rigidly rotate about the z-axis with constant angular frequency ω.



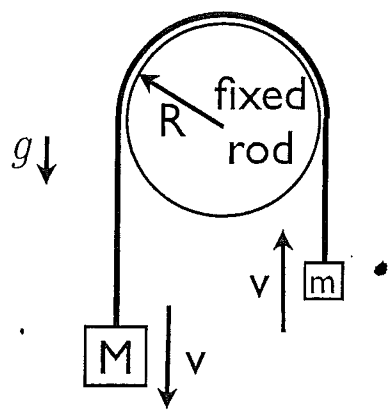
(a) Derive the equation of motion of the bead in terms of its radial distance $r(t)$ from the z-axis.

(b) Find the equilibrium points r_{eq} of the radial motion (*i.e.*, motion where $r(t) = r_{\text{eq}}$, constant),

(c) Find the frequencies of small radial oscillations about those equilibrium points that are stable.

2. **Rope friction**

A heavy mass M and a light mass m are joined by a rope which is passed across a fixed horizontal cylindrical rod of radius R, so the masses hang downwards under the action of gravity g. The heavy mass M has been carefully chosen so it falls with a constant speed v (and the light mass rises with the same speed). The mass of the rope is negligible.

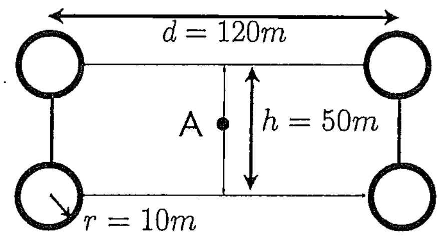


(a) What information about the coefficients of friction between the rope and the cylinder can be deduced from this observation?

3. **"Space hotel" (rotating space station with artificial gravity)**

In the design of a space station to be used as a "space hotel" for astronauts to live in, it is proposed to make it spin to produce an "artificial gravity" effect.

The "hotel" consists of four "living-quarter pods" which can each be modeled as a hollow spherical shell with a mass $m = 3\times10^4$kg and a radius $r = 10$m. These are coupled together with rigid beams of negligible mass so that their centers are at the vertices of a $d \times h = 120\text{m}\times50\text{m}$ rectangle.



(a) Compute the moments of inertia I_1, I_2, I_3 (relative to the center of mass A of the structure) for the three orthogonal principal axes of the moment-of-inertia tensor, chosen so that $I_1 < I_2 < I_3$. Identify these three axes on a sketch of the "hotel". (Note that $5^2 + 12^2 = 13^2$.)

In the design, the "hotel" rotates about the I_2 principal axis with a frequency f_g so that an astronaut at the center of a pod experiences the equivalent of 1/5 of the Earth's gravity at sea level. This "centrifugal force" is directed away from the axis of rotation.

(b) What is the value of f_g required to generate this apparent gravity?

The "hotel" is constructed, and made to rotate as described above. Unfortunately, the connecting beams are slightly flexible, which allows damped small vibrational modes of the structure to become excited and dissipate rotational kinetic energy as heat, while conserving the angular momentum $\vec{L}$ of the rotating hotel.

(c) Estimate the characteristic time scale for the exponential growth of ω_3, the angular frequency of rotation about the axis with moment of inertia I_3. Explain why the rotational motion about the axis with moment of inertia I_2 is not stable with respect to dissipation of the total rotational kinetic energy.

Section B. Electricity and Magnetism

1. **Birefringent material**

A "birefringent" (doubly-refracting) crystalline material is a material with an anisotropic dielectric tensor, so its index of refraction depends on the plane of polarization of incident electromagnetic waves.

For a certain material, the dielectric tensor is given by

$$\epsilon_{ij} = \epsilon \begin{pmatrix} 1 & \Delta & 0 \\ \Delta & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

where ϵ, Δ are real, and $0 < \Delta < 1 - \epsilon^{-1}$, which means that the constitutive relations of the material are

$$D_i = \epsilon_0 \sum_j \epsilon_{ij} E_j, \quad B_i = \mu_0 H_i,$$

where ϵ_0 and μ_0 are the vacuum permittivity and permeability.

(a) Find the two possible frequencies $\omega_\nu(k)$ and corresponding polarization directions (of $\vec{E}_0$) of a plane wave

$$\vec{E} = \vec{E}_0 \cos(kz - \omega_\nu(k)t)$$

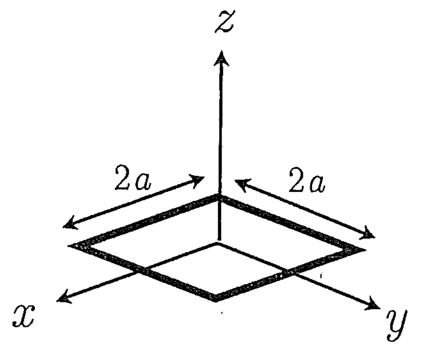
propagating parallel to the z-axis inside the material with a given wavevector $\vec{k} = (0, 0, k)$.

(b) The plane $z = 0$ is the surface between the vacuum $(z < 0)$ and the birefringent material $(z > 0)$. A plane wave with wavenumber $\vec{k} = (0, 0, k_0)$, frequency ω, and with linear polarization along the x-axis, coming from the vacuum, is normally incident on the surface of the birefringent material. What is the direction of polarization of the reflected wave?

Note: you may find it convenient to change to a different Cartesian coordinate system in which the dielectric tensor is diagonal.

2. Square loop antenna

A square conducting loop with sides of length $2a$ is in the x-y plane, concentric with the origin, with its sides parallel to the coordinate axes. The current in the loop is $I_0 \cos(\omega t)$, where $a \ll \omega/c$. The loop is surrounded by vacuum.



(a) What is the electric field in the radiation zone (*i.e.*, in the limit of distances $r \gg \omega/c$ from the origin)? Characterize the type of radiation this represents.

(b) How does the total power P radiated by the loop depend on ω? What is its precise value?

An infinite, perfectly-conducting plane is now placed at $z = -b$, where $b \ll c/\omega$.

(c) What type of radiation field is now seen for $r \gg \omega/c$, and how does its radiated power depend on frequency in this case?

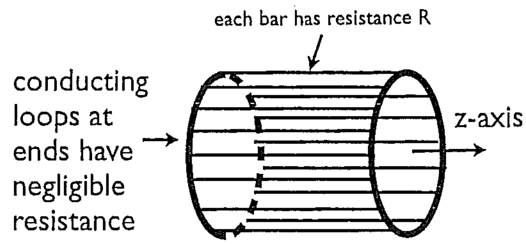
(Note that you are only asked about the **frequency-dependence** of the power, not for an elaborate calculation which would be needed to find the precise value of the radiated power in (c)).

3. **Torque produced by an AC induction motor.**

An induction motor consists of two components. The "*stator*" produces a time-dependent magnetic field with a direction that rotates with angular frequency ω_s, determined by the frequency of the AC current source. Take it to be

$$(B_x(t), B_y(t), B_z(t)) = B_0(\cos\omega_s t, \sin\omega_s t, 0).$$

The second component is the *rotor*, which in one design resembles a cylindrical "cage" that is free to rotate about its axis (the z-axis), formed by $N \gg 1$ equally-spaced conducting bars of length ℓ, each with resistance R, connected by a metal ring of radius r at each end of the cylinder (which has negligible resistance).



Electrical currents in the cage may be described as N independent current loops, each one defined by two adjacent bars, connected by the rings at each end of the cage. Assume the self-inductance and mutual inductances of these loops are negligible (much smaller than R/ω_s). The positions of the bars are defined by the line segments

$$(x, y, z) = (r\cos\theta_j, r\sin\theta_j, z), \quad 0 < z < \ell, \quad j = 1, 2, \ldots, N.$$

with $\theta_j = 2\pi j/N + \omega_r t$ (the rotor rotates with angular velocity $\omega_r \neq \omega_s$).

(a) As a function of time, what is the induced emf across bar j?

(b) Find the time-averaged torque exerted on the rotor as a function of ω_r.

Section A. Quantum Mechanics

1. **Energy of a large atom**

This problem is about large neutral atoms with large nuclear charge Z. Treat the nucleus as an stationary, infinitely-heavy point charge, located at the origin in space. Take the atom to be neutral, so the number of electrons is $N = Z$. As an approximation, ignore relativistic effects (which limit Z to less than 137); also **neglect the electron-electron Coulomb repulsion** and consider only the attraction of the electrons to the nucleus.

(a) Explicitly, what is the Hamiltonian H for this problem, and what are the energies of its electronic orbitals?

(b) Give expressions for the ground state energy E_n, and for the number of electrons $Z_n = N = Z$, when exactly n "closed shells" of electronic orbitals are filled. From the leading asymptotic behavior of Z_n and E_n for large n, obtain the leading asymptotic behavior of $E(Z)$ for large Z.

(c) $E = K + V$, where K is the kinetic energy and V the potential energy. In terms of E, what are the expectation values of K and V?

(d) If r_i, $i = 1, \ldots, N$, are the radial coordinates of the electrons, an "average distance from the nucleus", may be defined by the expectation value

$$\frac{1}{r_{\text{av}}} = \langle \frac{1}{N} \sum_{i=1}^{N} \frac{1}{r_i} \rangle.$$

How does r_{av} depend on Z in the large-Z limit?

2. **ANHARMONIC OSCILLATOR**

A non-relativistic particle with mass m moves one-dimensionally in the potential

$$V(x) = \frac{1}{2}m\omega^2 x^2 + \lambda x^4, \quad \text{with } \lambda > 0.$$

Let $|\Psi_0(\lambda)\rangle$ be the ground state of this system, and $E_0(\lambda)$ be the ground-state energy. For small λ, the quartic term in the potential can be treated as a small perturbation of the $\lambda = 0$ harmonic oscillator problem, which has an oscillation frequency ω.

(a) The particle coordinate x can be expressed as an operator in terms of $a^\dagger$ and a, the raising and lowering operators for the $\lambda = 0$ harmonic oscillator problem, where $a|\Psi_0(\lambda = 0)\rangle = 0$. Give such an expression for x.

(b) Compute the perturbation expansion of the ground-state energy $E_0(\lambda)$ up to first order in λ.

(c) Again up to first order in λ, compute the perturbation expansion of the ground-state expectation value $\langle\Psi_0(\lambda)|x^2|\Psi_0(\lambda)\rangle$.

In the opposite limit of *large* positive $\lambda \to \infty$, the leading behavior of the ground state energy $E_0(\lambda)$ will be proportional to λ^α where α is a positive exponent.

(d) (Up to an undetermined numerical multiplicative factor) find the asymptotic large-λ behavior of the ground-state energy $E_0(\lambda)$, giving the explicit value of α.

(You may find a simple variational estimate of $E_0(\lambda)$, or scaling arguments, helpful in part (d).)

3. **Hyperfine Splittings**

An isolated hydrogen atom has a hyperfine interaction between the spins $\vec{S}_1$ of the proton and $\vec{S}_2$ of the electron of the form $J\vec{S}_1 \cdot \vec{S}_2$, $J > 0$. The two spins also have magnetic moments $\alpha\vec{S}_1$ and $\beta\vec{S}_2$, and a uniform static magnetic field $\vec{B}$ is present. Assume that the electron is in its $1s$ orbital ground state, and ignore the effects of orbital motion.

(a) Find the exact energy eigenvalues of this system and sketch the hyperfine-splitting spectrum as a function of magnetic field.

(b) In the basis of states $|S_1^z, S_2^z\rangle$, find the eigenstates associated with each level.

Section B. Statistical Mechanics and Thermodynamics

1. **One-dimensional model for rubber**

To model rubber, consider a polymer of N molecules of length a, connected end to end. One end of the molecular chain is fixed at $x = 0$. Assume that the molecules can only be oriented parallel to the x-axis, and that all possible configurations of the chain have the same internal energy. The chain is kept at fixed temerature T by contact with a heat-bath.

In this model, the two ends of molecule n are on the x-axis, at x_{n-1} and x_n, with $x_n - x_{n-1} = \pm a$, and $x_0 = 0$.

(a) What is the entropy of the chain when the free end is at $x_N =$ L?

(b) What is the tension in the chain when it is stretched to length L, with $L \ll L_{\text{max}} = Na$?

(c) For N large, how much work is required to stretch the chain from its equilibrium state with unconstrained length to a length $L \leq L_{\text{max}}$?

(d) During the stretching process, does the heat-bath absorb or give out heat? (Explain your answer).

2. **Thermionic emission of electrons from a metal surface**

Assume that, to escape from a metal, an electron coming from the interior must collide with the surface with enough momentum to overcome the confining potential that holds the electrons in the metal. Also assume that all electrons with such a momentum do escape.

(a) Calculate the flux (number per unit area per unit time) at room temperature T of electrons escaping from a metal with a work function ϕ that is of the order of an electron-Volt. Treat the electrons as a non-relativistic ideal Fermi gas.

3. **Van der Waals equation.**

The Van der Waals equation of state is

$$P = \frac{Nk_BT}{V - Nb} - a\frac{N^2}{V^2}$$

for the pressure P of a fluid of N interacting atoms in volume V at temperature T. This models the liquid-gas phase transition and its critical point.

(a) **Briefly** explain the physics represented by each of the two correction terms that Van der Waals added to the ideal gas equation of state (corresponding to the parameters b and a).

(b) The Van der Waals equation models the liquid-gas transition line in the phase diagram of a fluid, and its termination at a critical point. Calculate the parameters at the critical point: the critical pressure P_c, critical temperature T_c, and the critical density $n_c = (N/V)_c$.