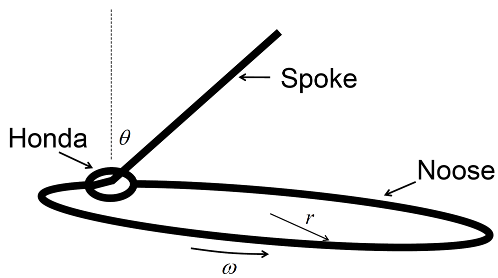
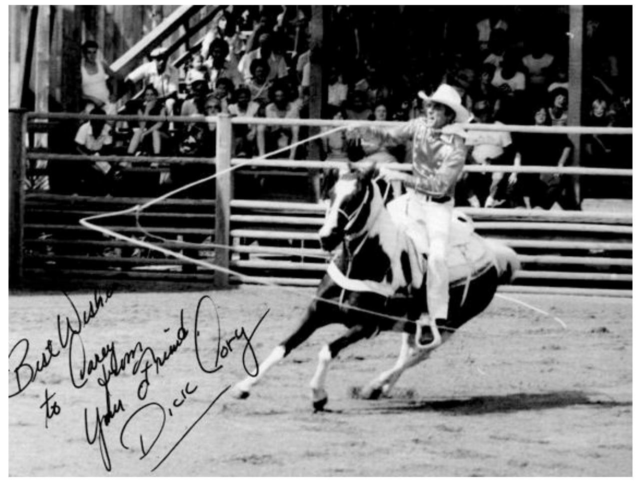
Section A. Mechanics

1. (Lasso)





Picture from: "The Lasso: a rational guide ..." ©1995 Carey D. Bunks

A ***lasso*** is a rope of linear mass density ρ that ends in a loop called the ***honda***. The free end of the rope is fed through the honda to create a large loop called the ***noose***. The remaining length of rope is called the ***spoke***, which is used to impart energy to the noose (spin it up) and to support it against the downward pull of gravity.

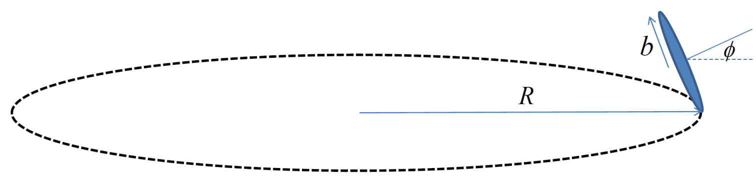
Consider the case of a circular noose of radius r spinning in a nearly horizontal plane with angular speed ω (you can neglect the tilt angle of the noose with respect to the horizontal). The spoke is supported above the center of the noose and makes an angle θ with respect to the vertical. There is no friction between the rope and the honda.

a) What is the tension in the rope within the loop?

b) The inward force of the spoke on the honda tends to perturb the shape of the noose near the honda (*as can be seen in Dick Cory's picture, above*). To counteract this effect, the honda is given an additional mass m_{h} (say, by wrapping it with a heavy metal wire). What should m_{h} be in terms of the given quantities in order to maintain the circular shape of the noose?

c) Suppose the spinning noose is subject to a small-amplitude transverse disturbance (a kink) that results in a wave propagating along the rope in the direction opposite to the rotation of the noose. What is the angular speed of the wave as viewed from a reference frame at rest with respect to the spinning noose? What does your answer imply about the motion of the kink as seen by an observer in the fixed frame?

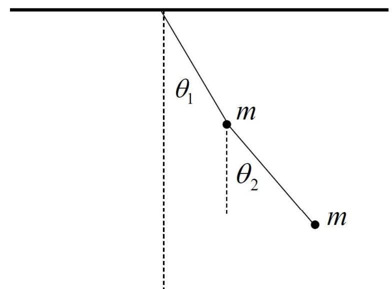
2. (Rolling Coin)



A coin (uniform solid cylinder) of mass M and radius b rolls without slipping on a horizontal table such that the axis perpendicular to its face makes a constant angle ϕ with respect to the table top (see diagram). The point of contact moves in a counter-clockwise (as viewed from above) circular path of radius R with constant linear speed v. What is the relationship between ϕ and the given quantities? In your solution, do **not** assume that ϕ is a small angle.

3. (Double Pendulum)

A double pendulum consists of two massless rigid rods of length L to which two identical masses m are attached (see diagram).



a) What are the characteristic frequencies of this pendulum for small-angle oscillations around equilibrium?

b) What are the normal coordinates for this system?

c) Describe the motion corresponding to each of the normal modes of vibration.

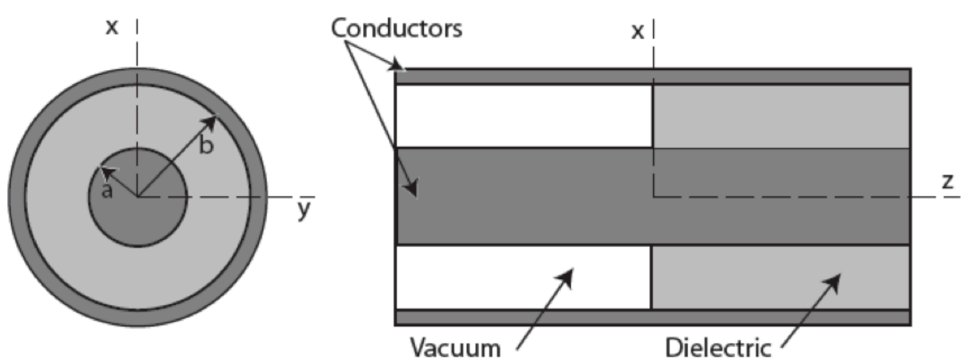
Section B. Electricity and Magnetism

1. Electromagnetic radiation in TEM mode propagates along a coaxial waveguide consisting of two concentric, right circular cylinders of infinite conductivity. The radius of the inner conductor is a, and of the inner surface of the outer conductor is b. The region of the conductors corresponding to negative values of z (the symmetry axis of the cylinders) is vacuum. The region at positive z is filled with a uniform lossless dielectric, (dielectric constant ε). The wave is propagating only in the positive direction in the dielectric, while there are incident and reflected waves in the vacuum region. Assume the incident wave has peak electric field E_o at the surface of the inner conductor, and oscillates with frequency ω.

   a) In terms of E_o , a, b, and any necessary constants for your system of units, find the time-averaged power of the incident wave propagating in the vacuum side of the cable.

   b) Calculate the electric and magnetic fields for the reflected, and transmitted waves. Specify the amplitudes, and space and time dependence.

   c)) Find the average force per unit area on the dielectric interface at $z = 0$.



2. A pulsed beam of charged particles is shot into a finite electrically isolated plate of ohmic conductance σ and dielectric coefficient ε. At the end of the pulse (at time $t = 0$) the charge per unit volume in the plate is non-uniform and given at $\mathbf{r}$ by $\rho_0(\mathbf{r})$, where the position vector $\mathbf{r}$ specifies points inside the plate. You may neglect any magnetic fields in the plate.

   a) Show that the final state of static equilibrium is one in which the charge is deposited only on the surface of the plate.

   b) Find the equation governing the charge distribution $\rho_0(\mathbf{r})$ for $t > 0$ as the system approaches static equilibrium.

   c) Solve this equation and show that the interior charge moves to the surface with a characteristic time constant τ. Determine the expression for τ.

3. A right circular cylinder of radius R and length L is carrying a uniform current I parallel to the axis of the cylinder.

a) What is the direction and magnitude of the magnetic field inside the cylinder? (Ignore end effects, and other sources of the B field).

b) Next, directed towards the above current-carrying cylinder and parallel to its axis is a parallel monochromatic beam of energetic charged particles. Show that within the following approximation the beam will be focused at a point after passing through the cylinder. Derive an expression for the focal length.

In the derivation neglect scattering and slowing down of the beam's particles due to interactions with the material within the cylinder (other than through the field described above), and make the thin lens approximation by: i) assuming that the cylinder is short compared to the focal length, yet at the same time, ii) ignoring end-effects.

c) Consider using the magnet to collect into a parallel beam antiprotons produced by a beam of high-energy protons that strike a target placed at the focal point of the magnetic lenss. Specifically: assume the magnet is a cylinder of lithium metal of length 15 cm and radius 1 cm, and the total current it carries is I. What current would be required to collect antiprotons that are produced with a momentum of 10 GeV/c at angles up to 50 mrad relative the beam axis?

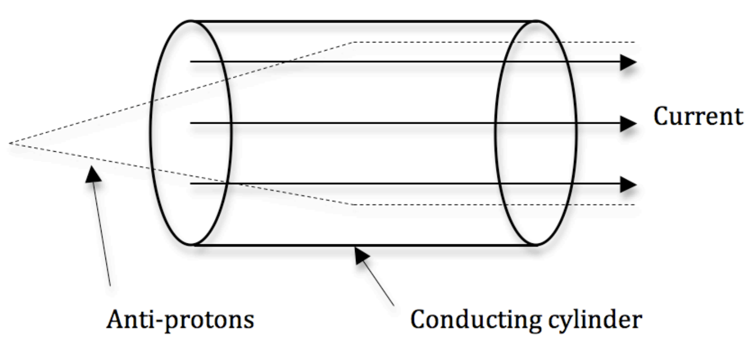


Fig. Ilustration for part c) (only)

Some useful constants, and magnitudes:
$\mu_0 = 4\pi \times 10^{-7}\, N\, A^{-2}$, $e = 1.60 \times 10^{-19}$ C, proton mass $m_P = 1.67 \times 10^{-27}$ kg.

Section A. Quantum Mechanics

1. (Quantum Zeno effect) A two-level quantum system is represented as a spin-$\frac{1}{2}$ entity, evolving under the Hamiltonian

$$H = \hbar \vec{\boldsymbol{B}} \cdot \vec{\sigma}$$

with $\vec{\boldsymbol{B}} = \{b, 0, 0\}$ pointing in the x direction, and $\vec{\sigma} = \{\sigma_x, \sigma_y, \sigma_z\}$ the Pauli spin matrices:

$$\boldsymbol{\sigma}_x = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad \boldsymbol{\sigma}_y = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}, \quad \boldsymbol{\sigma}_z = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}.$$

The system is prepared in the state $\sigma_z = +1$ at time $t_0 = 0$.

The following questions concerns the results of measuring the quantity σ_z at later times, assuming the measurement itself does not change the value of σ_z (though of course it cannot preserve the relative phase between the states $|\sigma_z = 1\rangle$ and $|\sigma_z = -1\rangle$).

a) What is the probability of finding the system in the flipped state, $\sigma_z = -1$, at time $t_1 > 0$?

b) The measurement is repeated at time $t_2 = t_1 + \tau$, after recording the value of σ_z at time t_1 (either $\sigma_z = +1$ or $\sigma_z = -1$). What is the probability that the system will be found at the state $\sigma_z = -1$ at t_1?

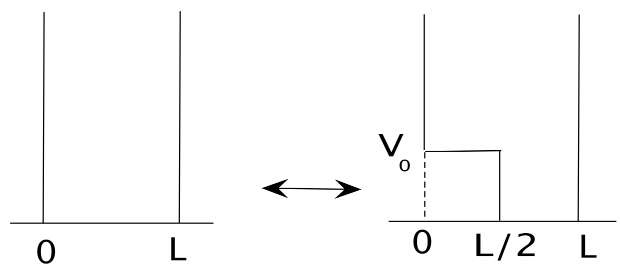
c) Next, re-starting from the state $\sigma_z = +1$ the observer begins to measure σ_z repeatedly at intervals $\tau \ll b^{-1}$, up to time T ($\gg \tau$), that is at times $t_n = n\tau$ with $n = 1, 2, 3, ..., T/\tau$. How small should τ be made in order to reduce the probability of finding the system at time T in the flipped state ($\sigma_z = -1$) to less than $p_0 = 0.1$?

2. A particle of mass m is confined to the interval $[0, L]$ by a one -dimensional infinite square well. It is initially in the ground state of the Hamiltonian with the confining potential.

   a) At time $t = 0$ the potential *within the well* is suddenly changed to:

   $$V(x) = \begin{cases} V_0, & \text{for } 0 < x < L/2 \\ \\ 0, & \text{for } L/2 < x < L \end{cases}$$

   with $V_0 \ll E_1 - E_0$ (the latter being the gap between the two lowest states of the initial operator).

   The perturbation lasts for time T, after which the potential is restored to its initial value. What is the probability that after the potential is restored the particle's energy is E_1, calculated to first order in $V_0/(E_1 - E_0)$?

   b) In a second experiment the value of V_0 (in the perturbing potential, as above) is increased very slowly, and to a much higher value $\overline{V} \gg E_1$. It is switched off instantaneously when that value is reached. What is the probability that at this point the particle will have the energy E_1.

3. The Hamiltonian for the spin degrees of freedom of Positronium in a magnetic field in the z-direction is given by

$$H \;=\; \alpha\,\vec{\sigma}^{(1)}\cdot\vec{\sigma}^{(2)} + \beta\,(\sigma_z^{(1)} - \sigma_z^{(2)})\;, \tag{1}$$

where α and β are constants and $\vec{\sigma} = (\sigma_x, \sigma_y, \sigma_z)$ denotes the vector of Pauli matrices (listed in problem QM1).

a) As a warm-up: describe the degeneracy of the energy levels of the Hamiltonian at $\beta = 0$.

b) Calculate the *eigenvalues* and describe (to the extent that you can without writing 'messy' expressions) the *eigenvectors* of H.

Section B. Statistical Mechanics and Thermodynamics

1. (Atmosphere's 'adiabatic lapse rate')

   This problem is intended to estimate how rapidly the atmosphere cools as you climb a mountain, modeling the atmosphere by an ideal gas which is in equilibrium with respect to adiabatic convection. The gas is of molar weight M (that is, 1 mole of the gas has the mass of M grams), and its molecules are of f degrees of freedom. At the surface of the earth ($z = 0$) the gas is at pressure P_0 and temperature T_0; the gravitational acceleration is g and the ideal gas constant is R.

   a) Derive the relationship between the temperature T of the gas and the pressure P, for an adiabatic flow of the gas (of f degrees of freedom per molecule).

   b) Assuming the gas is in mechanical equilibrium (that is, at each height z there is no net force on a slab of the gas of thickness dz) derive $\frac{dP}{dz}$.

   c) Use parts (a) and (b) to compute the dry adiabatic lapse rate $\frac{dT}{dz}$ (the change in temperature with altitude) in terms of M, g, R and f.

   d) Derive an expression for the pressure P as a function of the height z, for an adiabatic atmosphere.

2. (Hydrogen's two spin isomers)

   Molecular hydrogen H_2 consists of two protons and two electrons. Protons are fermions with spin$= 1/2$. The spins of the two electrons are paired so that the net electronic spin is 0, but the nuclear spins of the protons can be in either the total $S = 1$ spin state or the $S = 0$ state.

   a) Write down the partition function for the two nuclear spin states of H_2. Assume the energy difference between the $S = 1$ and $S = 0$ states is ΔE, with $S = 0$ the lower one.

   b) At $T = 300$ K, hydrogen is 70% $S = 1$ and 30% $S = 0$. Calculate a value for ΔE from this data.

   c) The latent heat ΔL of hydrogen at its boiling point (20.3 K at 1 bar) is 445 kJ/kg. The conversion of $S = 1$ to $S = 0$ H_2 is very slow (measured in days), and it is quite possible to quickly cool hydrogen gas to its boiling point with the 300 K nuclear spin distribution intact. Assume $\Delta E = 10^{-21} J$. Determine if liquid H_2 will entirely boil off when the spin conversion to the ground state occurs. Assume perfect thermal insulation, and that the hydrogen's nuclear spins convert completely to their ground state.

3. We have had a cold December, and it is time for ice skating. The Clausius-Clapeyron equation describes the slope, $\frac{dP}{dT}$, of the 1^{st}-order phase transition line in the pressure-temperature (P,T) phase diagram.

a) Derive the Clausius-Clapeyron Equation for $\frac{dP}{dT}$ in terms of the specific heat and the density difference between the two phases.

b) For the phase change of ice to water, the latent heat of fusion L is about +3$\times$10^5 J/kg, and the volume change ΔV is about -10^{-4} m^3/kg. Estimate the pressure needed to depress the freezing point of ice by 5 C.

c) Comment quantitatively on the urban legend that skates glide with low friction over ice because the ice melts under the pressure of the skate blade pressing down on it. Assume the skater is of mass 70 kg, the skate blade is 30 cm long and 5 mm wide, and the temperature is -5 C.