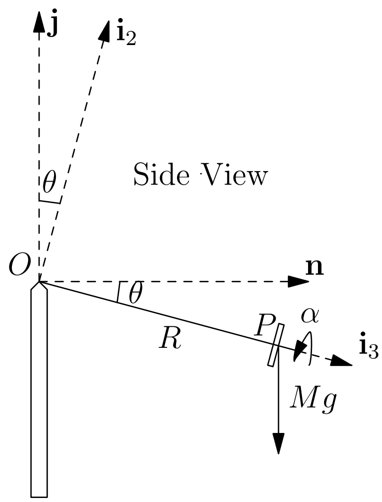
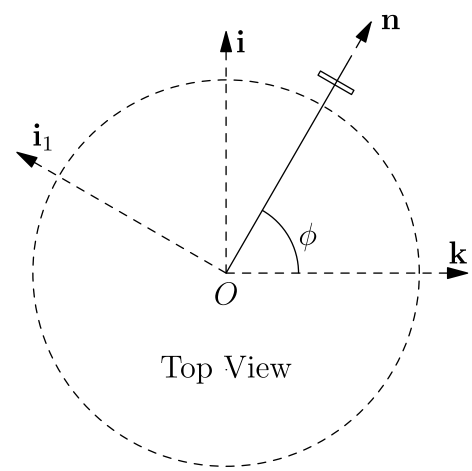
J06M.1 - Gyroscope

Problem

A gyroscope, illustrated in the figures below, is free to pivot about point O under the effect of gravity. Its total mass is M and its center of mass is located at point P at a distance R from O. In the reference frame $(O; \boldsymbol{i}_1, \boldsymbol{i}_2, \boldsymbol{i}_3)$ of the gyroscope (see figures), its moment of inertia tensor about point O is $\hat{I} = \begin{pmatrix} I & 0 & 0 \\ 0 & I & 0 \\ 0 & 0 & I_3 \end{pmatrix}$. If $(O; \boldsymbol{i}, \boldsymbol{j}, \boldsymbol{k})$ is the laboratory frame and $\boldsymbol{n}$ that axis at the intersection between the plane $\boldsymbol{i}_2\boldsymbol{i}_3$ and the plane $\boldsymbol{ik}$, define α to be the rotation angle of the gyroscope around $\boldsymbol{i}_3$, θ (the nutation angle) to be the angle between $\boldsymbol{i}_3$ and $\boldsymbol{n}$ and ϕ (the precession angle) as the angle between $\boldsymbol{k}$ and $\boldsymbol{n}$.





a) Write the Lagrangian of the system and its energy in terms of the angles α, θ, ϕ and of their time derivatives. [*Hint:* In order to find the expression for the kinetic energy $\frac{1}{2}\vec{\omega} \cdot \hat{I}\vec{\omega}$, first write $\vec{\omega}$ in the reference frame $(O; \boldsymbol{i}_1, \boldsymbol{i}_2, \boldsymbol{i}_3)$.]

b) Write the conservation laws for this system: energy and two projections of angular momentum.

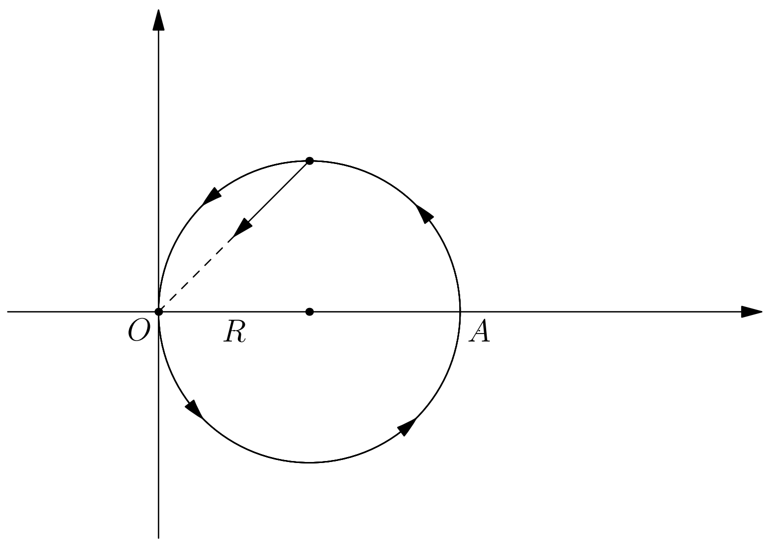
c) From the conservation laws deduce a closed equation for θ in the form $F(\dot{\theta}, \theta) = 0$.

d) At time $t = 0$ the gyroscope is place horizontally ($\theta = 0$) with zero nutation angular velocity ($\dot{\phi} = \dot{\theta} = 0$) and spin angular velocity $\dot{\alpha} = L_0/I_3$. Show that for $\theta \ll 1$ the previous equation and these initial conditions admit an approximate solution $\theta = \theta_0(1 - \cos\omega_n t)$. Compute the frequency ω_n, the amplitude θ_0, and the average precession velocity $\langle\dot{\phi}\rangle$. Find the condition on the initial data (i.e. on L_0) for which $\theta \ll 1$ remains a good approximation at all times.

J06M.2 - Displaced Circular Orbit

Problem

A point mass m is moving on a circular orbit of radius R under the effect of a central force directed toward the point O on the orbit (see figure below). Its speed at point A (A is diametrically opposite to O) is equal to v_A.



a) Find the expression for the force generating this motion.

b) Using the convention that the potential energy vanishes infinitely far from the center of attraction, compute the values of the energy and of the angular momentum for the circular orbit.

c) Find the time needed for the point mass to complete the orbit.

J06M.3 - Bead on a Rotating Wire (J04M.1)

Problem

A bead of mass m slides without friction on a wire. At time $t = 0$ the wire is in the x-z plane and has shape

$$z = a\left(\frac{|x|}{a}\right)^{\alpha},$$

with $a > 0$ and $\alpha > 0$. The wire is rotating about the z axis with a constant, nonzero angular velocity ω, without changing this shape. Earth's gravity causes a force mg on the bead along the $= \hat{z}$ direction.

a) Find the equation of motion of the bead.

b) For each value of $a > 0$ and $\alpha > 0$, find all solutions to this equation of motion where the bead is not moving with respect to the wire.

c) For each solution you found in part b), determine whether there are nearby solutions with the bead undergoing small amplitude *harmonic* motion with respect to the wire, and solve for the frequency of those oscillations when they exist.

d) For what values of $a > 0$ and $\alpha > 0$ are there solutions where the bead starts near the z axis with a finite speed, but then "escapes" to infinitely far from the axis? When this happens, what is the trajectory of the bead at long times (assuming the dynamics remains nonrelativistic)?

J06E.1 - Dielectric Cylinder in an Electric Field

Problem

An infinitely long cylinder of radius a and dielectric constant ϵ is placed in an initially uniform electric field of strength E_0. The axis of the cylinder is oriented at a right angle to the direction of the field.

a) Find the electric potential $\Phi(r, \theta, z)$ both inside and outside of the cylinder, in cylindrical coordinates (r, θ, z), where the z axis is the axis of the cylinder.

b) Find the electric fields $\boldsymbol{E}$ and $\boldsymbol{D}$ inside the cylinder.

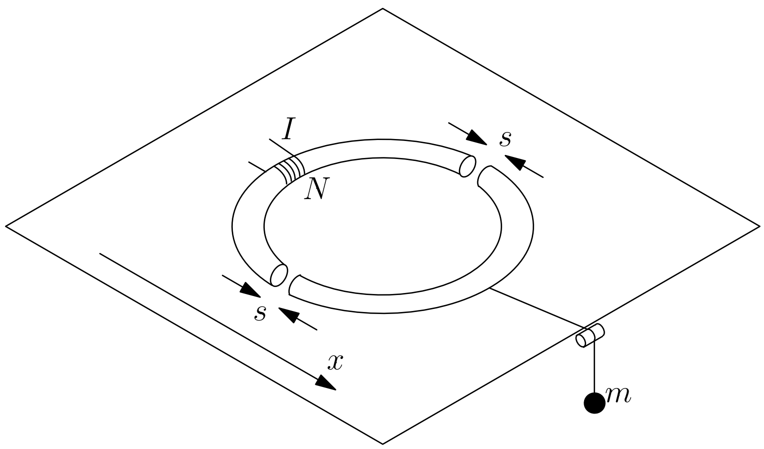
c) What is the surface polarization (bound charge) density σ_b at $r = a$? What is the volume polarization charge density ρ_b for $r < a$?

d) What is the electrostatic energy per unit length inside the cylinder?

J06E.2 - Half Rings of Magnetic Material

Problem

A ring with relative permeability $\mu_r = 400$, minor radius $a = 1.5$ cm, and major radius $R = 50$ cm is placed on an horizontal (x-y) plane. The ring is cut transversally at two diametrically opposite points with the same x coordinate; the first half-ring is fixed to the plane, while the second can slide frictionlessly along the x direction (see figure below). A current $I = 0.8$ A (kept constant by an external power supply) flows into a solenoidal coil with $N = 800$ turns tightly wound on the first half of the ring. A mass m can hang from am massless wire connected to the second half-ring.



The two half-rings are initially touching, then they are pulled apart to a separation of distance $s = 3$mm (see figure), and two small cylindrical pieces of wood (with relative permeability $\mu_r' = 1$) are inserted into the gaps. Compute, giving *numerical answers* for parts c) and d):

a) The magnitudes of the magnetic fields $\boldsymbol{B}$ and $\boldsymbol{H}$ as a function of the separation s for $s \ll R$, both within the rings and within the gaps. You may assume that the fields are uniform within the ring, and within the gap, and negligible elsewhere.

b) The total magnetic energy as a function of the separation s for $s \ll R$.

c) The self-inductance of the coil after the separation ($s = 3$ mm).

d) The minimum value of the mass m needed to pull the second half-ring away from the wooden cylinders.

J06E.3 - Harmonic Oscillator Radiation

Problem

A classical particle of mass m and charge q moves in an isotropic three-dimensional harmonic potential with "spring constant" K such that its trajectory is nearly circular at all times.

a) What is the characteristic time (time constant) for the decay of the kinetic energy of this system due to electromagnetic radiation?

b) What condition(s) must be satisfied so that the fraction of the energy radiated per period of the motion is small (i.e. so that the the quality factor of this oscillator remains high), and hence the trajectory is indeed nearly circular?

c) Verify that this requirement implies that the radiation-reaction force is small compared to the spring force on the particle.

J06Q.1 - Position and Momentum

Problem

a) For a particle moving in three dimensions with Hamiltonian

$$H = \frac{p^2}{2m} + V(\vec{r}).$$

in an arbitrary quantum state, what are the time derivatives, $d\langle\vec{r}\rangle/dt$ and $d\langle\vec{p}\rangle/dt$, of the expectation values of the position and momentum?

b) For times $t < 0$, a one-dimensional simple harmonic oscillator with mass m and frequency ω is in its ground state, at energy $\hbar\omega/2$, with $\langle x\rangle = 0$. At time $t = 0$ a uniform electric field E is instantaneously turned on and remains on for $t > 0$; it couples to the particle's charge q. What is the full time- and x-dependence of the particle's wave function $\psi(x, t)$? If its energy is measured at time t, what are the possible results and their probabilities?

J06Q.2 - Two Indistinguishable Bosons

Problem

Consider two indistinguishable nonrelativistic **bosons** of mass m, constrained to move one dimensionally around a circle of perimeter L (equivalently, on a line of length L with periodic boundary conditions). The two particles interact via a potential that is a delta-function, $V(x_1, x_2) = g\delta(x_1 - x_2)$. This interaction may be of either sign and of any strength. Give answers for all values of g, including $g = 0$.

a) The particles have spin zero. Solve for the wave function, energy and degeneracy of the ground state(s). Some of your answers here and below may involve a parameter that you may define as the solution to a transcendental equation.

b) Now the particles have spin one. The interaction is spin-independent. Again, find the wave function(s) (including the spin component), energy, degeneracy and total spin of the ground state(s). **Also**, find the total spin and the degeneracy of the lowest-energy excited state(s) for each value of g.

J06Q.3 - Magnetic Resonance

Problem

A particle of spin 1/2 and magnetic moment μ is at rest in the time-dependent magnetic field

$$\vec{B} = B_0\hat{z} + B_1\hat{x}\cos\omega t - B_1\hat{y}\sin\omega t,$$

which is often employed in magnetic resonance experiments. If the particle has the z component of its spin up (pointing along the positive z direction) at time $t = 0$, what is the probability that a measurement will find the z component of its spin down at time $t > 0$?

J06T.1 - Bose Einstein Condensation

Problem

A spin-zero particle of mass m moves nonrelativistically in the 3-D harmonic potential given by

$$V(x, y, z) = \frac{m\omega^2}{2}(x^2 + y^2 + z^2).$$

a) Obtain an expression for $D(\epsilon)$, the density of states for this particle, that is valid at energies much larger than $\hbar\omega$, where the energy ϵ can be approximated as a continuous variable.

b) Suppose there are now N (where N is large) noninteracting spin-zero particles in this harmonic oscillator potential. The particles are in equilibrium at temperature T, with $k_B T \ll \hbar\omega$. What is the chemical potential of the system in this low T regime (including the leading dependence on N for large N)?

c) In the thermodynamic limit of large N, this system has a Bose-Einstein condensation (BEC) such that the number of particles in the ground state is large even for temperatures well above $\hbar\omega$. The number of particles in the ground state is $N_0(T) = N(1 - (T/T_E)^\alpha)$, where T_E is the Einstein condensation temperature. Determine the exponent α and an expression for T_E. You may encounter a dimensionless integral whose value is not readily evaluated; you may give your answers in terms of this integral. For T_E to remain finite in the thermodynamic limit, the trap must be "softened", so that ω decreases in the appropriate way as N grows in order to keep T_E finite in the limit.

J06T.2 - Liquid-Gas Phase Transition

Problem

Consider the following approximate equation of state describing a liquid-gas phase transition and critical point:

$$p(V - Nb) = Nk_BT \exp(-Na/(k_BTV)),$$

where a and b are constants (not necessarily the same constants that appear in the usual van der Waals equation of state).

a) Breifly describe the physical meaning of the constants a and b.

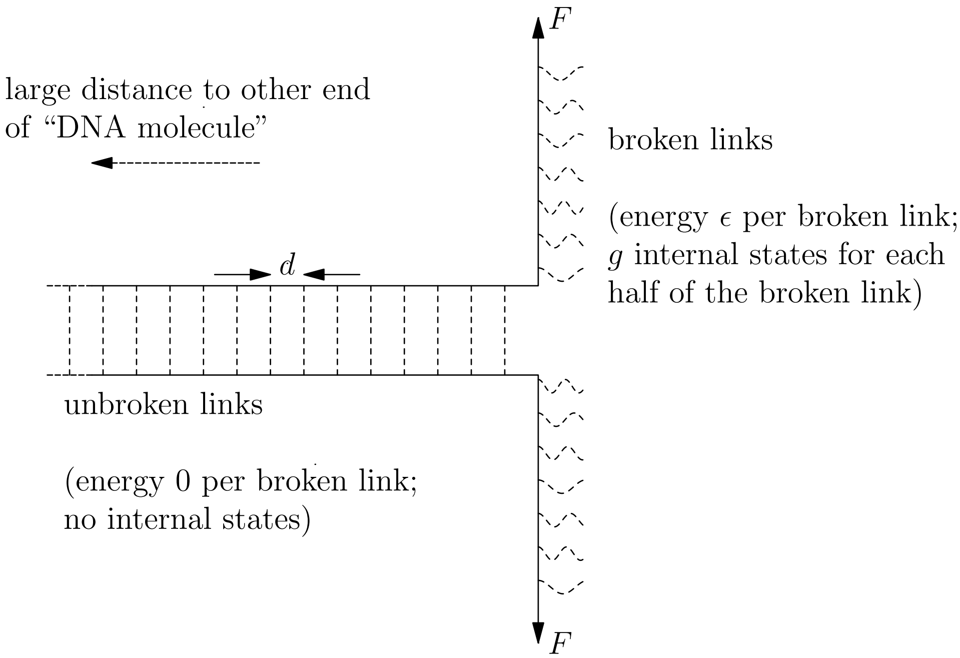
b) On a p-V diagram sketch some representative isotherms for this gas, showing all qualitatively important effects. Be sure to show and clearly label the critical isotherm, at least one isotherm at a higher temperature than the critical isotherm and at least one isotherm at a lower temperature than the critical isotherm. Show qualitatively correct equilibrium isotherms **after allowing for phase separation; do not show any metastable or unstable states**.

c) Determine p_c, V_c and T_c, that is, the pressure, volume and temperature at the critical point, given the above equation of state.

J06T.3 - DNA Molecule

Problem

A simple "toy model" model for how complementary strands of DNA are bound together resembles a zipper (see figure). The two strands are connected by "links" (base pairs) spaced at equal intervals d along the strands. It costs an energy ϵ to break a link, and a link can only be broken if its neighbor to the right is also broken. An unbroken link is a unique internal state, but each of the two dangling ends of a broken link can be one of g internal states.



At the right-hand end of the DNA molecule, the experimenter applies a tension force F to each of the two strands to try to separate them. This force is not strong enough to separate the chains at $T = 0$.

a) Assume that $g = 1$ (so broken links have no internal states). At finite temperatures $k_B T \gg \epsilon$, what is the mean number $\bar{n}$ of broken links near the ed of the DNA molecule, when $F = 0$? (Assume that $\bar{n}d$ is much smaller than the total length L of the DNA molecule.) How does it change when the force is applied?

b) Now assume that $g > 1$. Write down the configurational partition function, and obtain the free energy associated with the links between the strands. Obtain the critical temperature $T_c(g, \epsilon, F, d)$ above which the two strands of an infinitely long DNA molecule would be pulled apart by the applied force F.

c) Obtain an expression for $\bar{n}(T, g, \epsilon, F, d)$ valid for an infinitely-long DNA molecule at all temperatures less than T_c (including $k_B T \ll \epsilon$), and make a sketch showing its principal features.