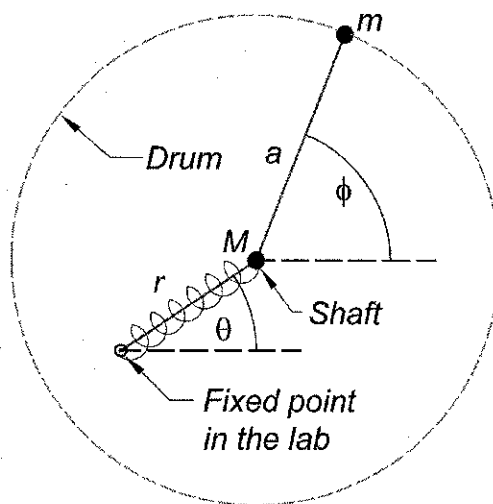


Section A. Mechanics

1. A rigid straight beam of mass M rests on and is supported by N exactly equivalent posts. The posts are located at positions $x = x_i$ along the beam ($i = 1, \dots, N$), with the center of mass of the beam at $x = 0$. The posts are very stiff, but compressible, obeying Hooke's law. The positions x_i of the posts are arbitrary, but assume that all posts do bear a nonzero load and that the tops of the posts are all at precisely the same height in the absence of the load.
 - a) Find the expression for the compressive force on each post in static equilibrium in terms of M , N , the $\{x_i\}$ and/or g .
 - b) Consider the modes of vibration of this system that involve vertical motion. Derive the condition on the set $\{x_i\}$ under which the normal modes of vibration consist of one mode where the beam moves vertically but does not rotate and another mode where it rotates with no motion of its center of mass.

2. **Washer.** Consider a top-loading washing machine consisting of a vertical cylindrical drum of mass M and moment of inertia I around an axis along its shaft, about which it rotates. The washing machine load is not balanced, and we approximate this asymmetric load as a single added point mass m located at a fixed radius a from the shaft, and at a fixed azimuth relative to the drum. The figure shows a top view of the washer.

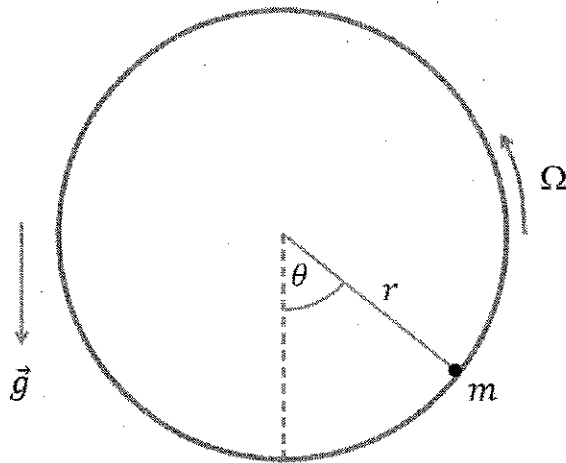
The shaft remains vertical, but otherwise its position is not fixed in the lab (laundromat) frame. Rather, a set of massless springs with effective force constant k connect the shaft to a fixed support, taken to be the origin of the system. The position of the shaft is then described by cylindrical coordinates (r, θ) in the lab frame, and the springs have effective relaxed length zero ($r = 0$ is the static equilibrium position). The azimuth of the line from the shaft to the mass m is labeled ϕ , and the time derivative of ϕ is constrained to be constant by the washing machine motor, so $\dot{\phi}(t) = \Omega t$. The purpose of the spring system is to stabilize the motion of the drum in the presence of an unbalanced load.



- Write down the equations of motion for r and θ .
- There is a finite and stable steady-state solution to these equations with $\dot{r} = \dot{\theta} = 0$ for all Ω , except right on resonance. Find these solutions for Ω both below and above the resonance.

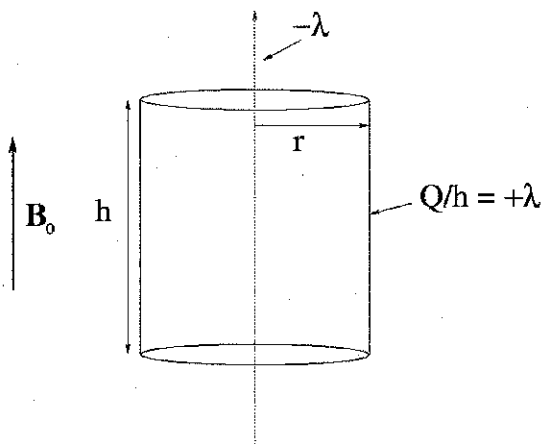
3. **Dryer.** A small load of mass m is sliding with friction coefficient μ along the interior wall of a cylindrical front-loading clothes dryer with radius r . The cylinder rotates with constant angular speed $\dot{\theta} = \Omega$ about its horizontal symmetry axis. The mass is initially in equilibrium at angle θ_0 as measured from the lowest point on the cylinder. If perturbed from equilibrium by a small angle ϵ , the mass executes simple harmonic motion around θ_0 with frequency ω .

Find expressions for θ_0 and ω in terms of the given quantities.



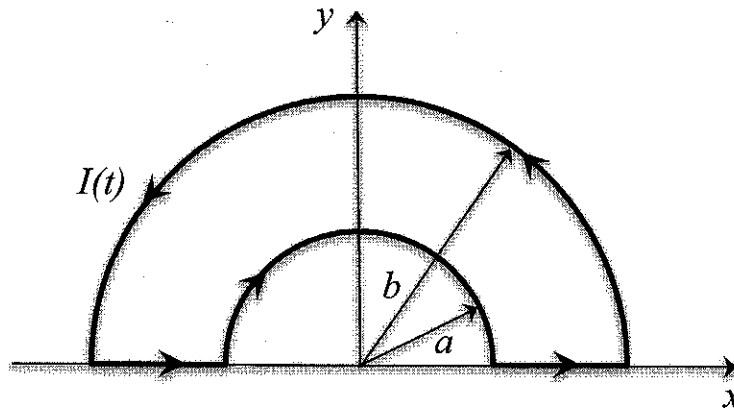
Section B. Electricity and Magnetism

1. A long thin non-conducting cylinder of radius r and height $h \gg r$ (figure not to scale) is concentric with a line charge of charge per unit length $-\lambda$. The cylinder has a uniform surface charge density with equal and opposite total charge per unit length $+\lambda$. The cylinder is free to rotate about its symmetry axis and has moment of inertia per unit length I/h . At times $t < 0$ the cylinder is at rest and a spatially uniform axial external magnetic field $B_0 \hat{z}$ is present, as shown in the figure. At time $t = 0$, the externally applied field is ramped down to zero.
 - a) Compute the torque on the cylinder in terms of $\frac{dB_z(t)}{dt}$, with $B_z(t)$ the (approximately uniform) axial magnetic field *within* the cylinder.
 - b) Find the angular velocity of the cylinder after the external field is reduced to zero, noting that the final field within the cylinder will be non-zero. Express your answer in terms of λ , r , B_0 , I , and/or h and whatever fundamental constants are required.
 - c) Recalling that the density of linear momentum stored in the electromagnetic field is proportional to the Poynting vector, express the *angular* momentum of the initial state. Demonstrate that the total angular momentum (mechanical plus electromagnetic) is conserved between the initial and the final conditions.



2. Thomson scattering is the scattering of light from a free electron (no binding, no damping). Derive the total cross section σ for Thomson scattering by treating the scattered radiation generated by an electron in an electromagnetic plane wave. Assume that the speed of the electron remains small: $v \ll c$.

3. A piece of wire is bent into the loop shown in the figure, consisting of semicircles and straight segments. It carries a current that increases linearly with time, $I(t) = kt$, and oriented in the direction indicated. Find the electric field $\vec{E}(t)$ at the center point $x = y = 0$ (exact solution).



Section A. Quantum Mechanics

1. Consider the scattering of a nonrelativistic quantum-mechanical particle of mass m from the finite spherical square well in three dimensions given by the radial potential: $V(r) = -V_0 < 0$ for $r < a$, while $V(r) = 0$ for $r \geq a$. Take the limit of zero incident energy, where the de Broglie wavelength λ of the incident particle satisfies $\lambda/a \rightarrow \infty$.
 - (a) In this limit obtain the differential cross section $\frac{d\sigma}{d\Omega}$ and the total cross section σ .
 - (b) This limiting zero-energy cross section diverges to ∞ at certain values of V_0 . What are those values of V_0 ? What is the physical significance of such divergences of σ ?

2. Tritium (${}^3\text{H}$, a radioactive isotope of hydrogen) decays to ${}^3\text{He}$ with the emission of an electron (and an antineutrino). Assume that this decay and electron emission is rapid enough so that as far as the other electron is concerned all that happens is that the atom's nucleus instantaneously changes its charge from $+e$ to $+2e$.
- a) Write the normalized ground state wave function for the one-electron atom or ion with nuclear charge Ze , neglecting spin and other fine-structure or relativistic effects. You may take it as given that the wave function is of the form $\psi(r, \theta, \phi) = A e^{-\kappa r}$ with A and κ dependent on the relevant parameters.
- b) Assuming the tritium atom was originally in its ground state, what is the probability of finding, immediately after the decay, the resulting He^+ ion in its ground state?
- c) In the event that the resulting He^+ ion is not in its ground state, compute its average excitation energy relative to the ion's ground state.

3. A system of two massive particles, of spins $s_a > 0$ and $s_b > 0$, is governed by the Hamiltonian:

$$H = K + V(|\vec{r}_a - \vec{r}_b|) + f(|\vec{r}_a - \vec{r}_b|) (S_z^{(a)} - S_z^{(b)}) ,$$

with K the usual kinetic energy operator and $\vec{S}^{(a)}$ and $\vec{S}^{(b)}$ the spin operators. The function f is negative at all distances: $f(r) < 0$, and the interaction potential $V(r)$ is finite and sufficiently attractive so that the system has at least one bound state. Let $\vec{S} = \vec{S}^{(a)} + \vec{S}^{(b)}$ be the total spin operator.

- a) Explain why this system's ground state is non-degenerate.
- b) What are the ground-state *expectation values* of the total spin's component S_z , and of the total spin operator $|\vec{S}|^2$? For which of these operators is the ground state also an eigenfunction?
- c) Consider now the case $s_a = 1$, $s_b = 1/2$. List the possible values of $|\vec{S}|^2$ and S_z . What are the probabilities of observing these outcomes when $|\vec{S}|^2$ and S_z are measured in the system's ground state?

Section B. Statistical Mechanics and Thermodynamics

1. Approximate the daily cycle of temperature T as a function of time t at the Earth's surface where the air is in contact with the ground as

$$T(t) = T_0 - T_1 \cos(\omega t) .$$

Assume the solid Earth below this flat horizontal surface is a homogeneous material with temperature-independent specific heat per unit volume c and thermal conductivity κ . We use the standard definition of thermal conductivity so the heat current density is given by

$$\vec{j}_Q = -\kappa \vec{\nabla} T .$$

- a) Calculate the temperature $T(t, z)$ at time t and depth z below the surface.
- b) Considering the daily cycle, assume the minimum and maximum air temperatures occur at 4 AM and 4 PM. At what times of day is the heat current zero at the surface (no heat flow between the air and the ground)?
- c) If the daily and yearly oscillations of air temperature both have the same amplitude T_1 (not true in Princeton, but it is roughly true at various locations), what is the ratio of the characteristic depths to which these two temperature signals penetrate the Earth?

2. We have two large solid blocks with heat capacities C_1 and C_2 . Assume these heat capacities are each constant in the range of temperatures considered in this problem. Initially the two blocks are at temperatures T_{1o} and T_{2o} and have entropies S_{1o} and S_{2o} , respectively, with $T_{1o} > T_{2o}$. In this problem there are no volume changes.
- a) Let these two blocks be in an isolated enclosure, so that no heat or work can flow in or out from the rest of the universe. What is the maximum total entropy that this system of two blocks can reach? How do you describe this maximum entropy state? Justify your answer.
- b) Alternatively, run a very small, reversible Carnot heat engine between the two blocks until equilibrium is reached and no more work can be extracted. What is the final temperature? How much work did the engine do?

3. Consider a classical one-dimensional magnet with Hamiltonian

$$H = -J \sum_{i=1}^N \vec{S}_i \cdot \vec{S}_{i+1}$$

where each \vec{S}_i is a classical (3-component) vector spin of fixed length S .

- a) Calculate $\langle \vec{S}_i \cdot \vec{S}_{i+1} \rangle$ at equilibrium at temperature T .
- b) Calculate the specific heat per spin $c(T)$ of this system in the limit $N \rightarrow \infty$.
- c) Consider the $T \rightarrow 0$ limit of part b). Is this consistent with the behavior of $c(T)$ for a **quantum** ferromagnet ($J > 0$) of spin S with this same Hamiltonian? If not, estimate (roughly) and state the correct quantum behavior of $c(T)$ for small T , explaining your reasoning. Ferromagnetic spin waves in this model have a frequency that depends on wavenumber k as $\omega(k) \sim k^2$ for small k .