Department of Physics, Princeton University

Graduate Preliminary Examination
Part I

Thursday, May 7, 2015
9:00 am - 12:00 noon

Answer TWO out of the THREE questions in Section A (Mechanics) and TWO out of the THREE questions in Section B (Electricity and Magnetism).

Work each problem in a separate examination booklet. Be sure to label each booklet with your name, the section name, and the problem number.

Section A. Mechanics

Rolling Sphere

1. A cylindrical pipe of interior radius R lies horizontally on the ground and remains stationary. A solid sphere of radius r is placed inside the pipe. What is the frequency of small oscillations of the sphere? Assume that the axis of rotation of the sphere is always parallel to the symmetry axis of the pipe, and the sphere rolls without slipping.

2. A particle of mass m moves under the influence of an attractive central force with potential $V(r)$.

   (a) Suppose the particle to move in a circular orbit with angular momentum ℓ. Derive the condition that determines how the orbit radius r_ℓ depends on ℓ.

   (b) Now consider a small perturbation $\delta r(t)$ around such a circular orbit. What is the condition that must be met for this perturbation to oscillate with a real frequency ω_ℓ (in which case the perturbation will not grow with time and the circular orbit will be stable)?

   (c) Now consider the special case $V(r) = -k/r^n$ with $n > 0$. For what values of n and ℓ are circular orbits stable (i.e. such that $\omega_\ell^2 > 0$)?

   (d) Are there any circumstances where the period of the circular orbit matches the period of small radial oscillations about the circular orbit. If so, what does this equality imply for the long-time trajectory of the particle?

Unicycle

3. One of the challenges of Jadwin Hall you no doubt have experienced is riding a unicycle down the ramp and not falling over as you do the 180° turn midway down the ramp. In this simplified study of unicycle stability, we assume that the total mass of the wheel plus rider is M, we assume the moment of inertia of the rider is zero, that the wheel has mass m, radius ℓ, and moment of inertia $I_w = m\ell^2$ about its axis, and that the center of mass H_{cm} of the unicycle and rider is at the top of the wheel. As you negotiate the turn, the unicycle moves with velocity V in a circular path of radius of $R >> \ell$.

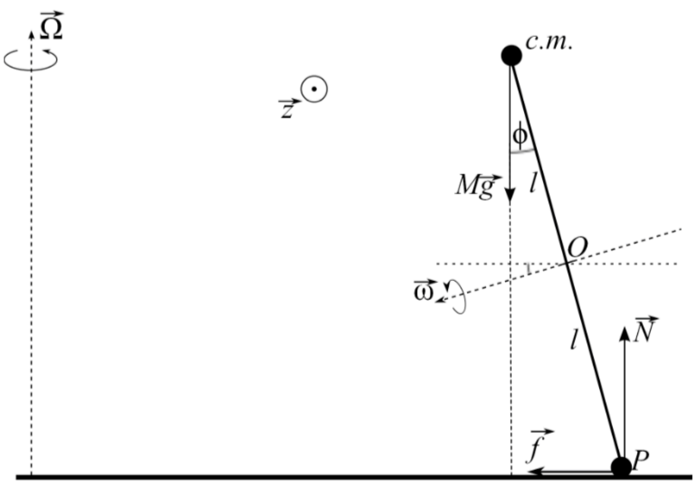


Figure 1: Coordinate system for the problem.

(a) Find the angle ϕ that the unicycle must lean through in the turn.

(b) Put in some typical numbers and and discuss the role of the angular momentum of the wheel in the stability of the unicyle. You can probably do this part successfully without the correct answer to (a).

Section B. Electricity and Magnetism

Radiating charge

1. A magnetic field $\mathbf{B}(x)$ points in the $\hat{z}$ direction in half of space as follows.

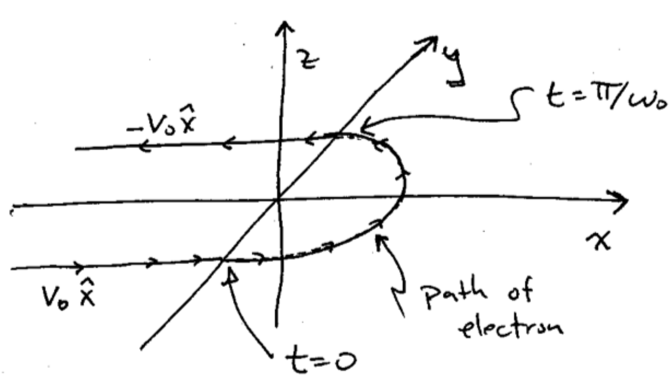
$$\mathbf{B}(x) = 0 \quad \text{for} \quad x < 0 \tag{1}$$
$$\mathbf{B}(x) = B_0\hat{z} \quad \text{for} \quad x > 0 \tag{2}$$

There is no electrostatic field. A non-relativistic electron of mass m_e initially moves at constant velocity $\mathbf{v} = v_0\hat{x}$ at $y = -a$ in the xy plane in the space with $x < 0$. As shown in the figure, at $t = 0$ the electron enters the space $x > 0$, travels in a semi-circle of radius a with an angular frequency of $\omega_0 = eB_0/m_e$ for a time π/ω_0 and then exits the region of field with $\mathbf{v} = -v_0\hat{x}$ at $y = a$.

(a) What is the differential power per unit solid angle, $dP(t)/d\Omega$, that an observer on the x-axis at a distance $x = r\hat{x} >> c/\omega_0$ measures when observing the charge? Sketch $dP(t)/d\Omega$ as a function of time and quantitatively label the duration, amplitude, and key features. [Hint: A lot of progress can be made using a simple model of how a charge radiates.]

(b) For an observer on the z-axis at a distance $z = r\hat{z} >> c/\omega_0$, again sketch the $dP(t)/d\Omega$ he or she would measure as a function of time. Quantitatively label the duration, amplitude, and key features.

(c) What is the total energy emitted by the electron?

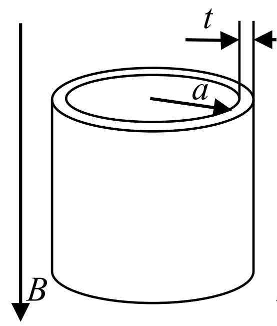


2. A thin sheet of material with ϵ_r =1 lies in the xy plane and carries a periodic surface charge density given by $\sigma(x) = \sigma_0 \cos(kx)$.

   (a) What is the electric field in the region $z > 0$? [Hint: $\nabla^2 V = 0$ in free space.]

   (b) A small neutral object of polarizability α is placed at $\mathbf{r} = (0, 0, z_0)$. What is the force (magnitude and direction) on it?

3. Consider shielding of a uniform oscillating magnetic field $B = B_0 \cos(\omega t)$ by a long open metal cylinder with its axis parallel to the field. The cylinder has electrical conductivity σ, radius a, height h, and wall thickness t, with $t \ll a$. Find the amplitude of the oscillating magnetic field inside the cylinder. Ignore edge effects and assume $\omega << h/c$.



Department of Physics, Princeton University

Graduate Preliminary Examination
Part II

Friday, May 8, 2015
9:00 am - 12:00 noon

Answer TWO out of the THREE questions in Section A (Quantum Mechanics) and TWO out of the THREE questions in Section B (Thermodynamics and Statistical Mechanics).

Work each problem in a separate booklet. Be sure to label each booklet with your name, the section name, and the problem number. Calculators are not permitted.

Section A. Quantum Mechanics

1. A particle of mass m, is moving on a line under the action of the Hamiltonian:

$$H_L = \frac{1}{2m} p^2 - b^2 \left[\delta(x+L) + \delta(x-L) \right]$$

(whose potential features a pair of attracting delta functions).

(a) State the symmetry, and sketch the shapes of the ground state and of the first excited bound state, in case such a state exists.

(b) Determine the minimal L_0 such that for $L > L_0$ the Hamiltonian has more than one bound state.

(c) What is the maximal number of bound states that H_L can have?

2. A point particle of mass m and electric charge q moves in a 3d harmonic oscillator potential with frequency ω and a uniform electric field of strength $\mathcal{E}$ pointing in the z-direction. The Hamiltonian is

$$H = \frac{\vec{p}^2}{2m} + \frac{m\omega^2\vec{r}^2}{2} - q\mathcal{E}z\,.$$

(a) What are the eigenenergies of this Hamiltonian?

(b) Find an expression for the ground state wave-function.

Assume now that the system is described by the above Hamiltonian only for $t < 0$, and that at $t = 0$ the electric field is suddenly turned off. For $t < 0$, the system is in its ground state.

(c) What is the probability that the system will end up in the new ground state right after the electric field is turned off?

(d) What is the expectation value of the electric dipole moment $\vec{d} = q\vec{r}$ at some given time $t > 0$?

3. Consider the scattering of a spinless particle of charge e and mass m in the screened Coulomb potential (also known as the Yukawa potential),

$$V(r) = -\frac{Ze^2}{r}e^{-r/a}.$$

Compute the differential scattering cross section, $d\sigma/d\Omega$, for fast incident particles in the appropriate approximation. What is the range of validity for your approximation?

Section B. Statistical Mechanics and Thermodynamics

1. Consider a system of noninteracting spin trimers. Each trimer is a set of three spins each of which can be either "up" or "down." Each trimer is described by the Hamiltonian

$$\hat{H} = -J\big(\sigma_1\sigma_2 + \sigma_2\sigma_3 + \sigma_3\sigma_1\big) - \mu_0 H\big(\sigma_1 + \sigma_2 + \sigma_3\big) \ . \tag{1}$$

Here J is the spin coupling energy and H is an external magnetic field. The individual spin polarizations σ_i are two-state Ising variables, with $\sigma_i = \pm 1$.

(a) Find the single trimer partition function Z_1.

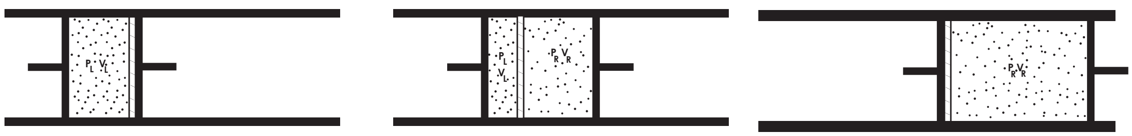
(b) Find the magnetization per trimer $m = \mu_0 \langle \sigma_1 + \sigma_2 + \sigma_3 \rangle$.

(c) Suppose there are $N_\triangle$ trimers in a volume V. The magnetization density is $M = N_\triangle m/V$. Find the zero field susceptibility [Hint: the definition of susceptibility is $\chi(T) = (\partial M/\partial H)_{H=0}$.]

(d) Find the entropy $S(T, H, N_\triangle)$.

(e) In the three limits $J \to +\infty$, $J \to 0$, and $J \to -\infty$ the expression for the entropy simplifies considerably. Give a simple physical explanation for each of these simplifications.

2. The figure below shows a throttling process in which two pistons "push" and "pull" gas through a porous divider (shown as a lightly hatched line) that is fixed inside a thermally insulated cylinder. No heat flows into or out of the cylinder. Initially, all the gas is on the left hand side of the divider and the right hand piston is up against the divider (top left plot). The top right figure shows an intermediate state and the bottom figure shows the final state. The pistons are moved in such a way that the pressure on the left hand side is always P_L and the pressure on the right side is P_R, with $P_R < P_L$. The final volume is larger than the initial volume.



(a) What thermodynamic potential has the same value at the end of the process as it did at the start? Prove it!

(b) If the gas is ideal, what is its change in the internal energy between initial and final states?

(c) Suppose now that the gas has a van der Waals equation of state

$$\left(P + \frac{N^2 a}{V^2}\right)(V - Nb) = NkT \tag{2}$$

where a and b are small numbers. The Helmholtz free energy is

$$F = -NkT\left\{\ln[n_Q(V - Nb)/N] + 1\right\} - N^2 a/V \tag{3}$$

where $n_Q = (mkT/2\pi\hbar^2)^{3/2}$. What is the internal energy U?

(d) What is the condition on the van der Waals coefficients a and b such that the gas cools on expansion into a very large volume in a throttling process? Qualitatively interpret the result. [Hint: $0 < Nb/V << 1$]

Adiabatic demagnetization

3. Consider a classical ideal gas of identical non-interacting particles of mass m in a container of volume V at initial temperature T_i. Let the particles have spin one-half and magnetic moment μ, and let the container be placed in a strong magnetic field H.

   (a) Compute the classical partition function for this system, taking proper account of particle identity. It may help you to know that the partition function for a single classical particle (ignoring the spin degree of freedom) is $Z_1 = n_Q V$ where $n_Q = (mkT/2\pi\hbar^2)^{3/2}$.

   (b) Calculate the total energy and entropy for this system

   (c) Now suppose that the container is thermally isolated and that the magnetic field is slowly reduced (i.e. adiabatically). Show that the temperature decreases continuously as H is decreased.

   (d) Show that if H is reduced all the way to zero the final temperature satisfies the inequality $T_i > T_f > 2^{-\frac{2}{3}} T_i$.