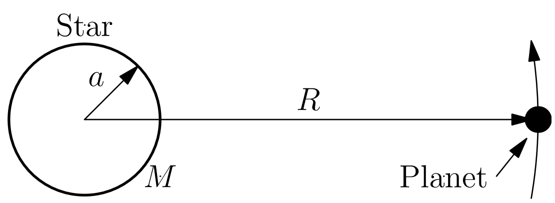
M00M.1—Precession of the Perihelion

Problem

With Newtonian mechanics, we wish to compute the rate of precession of the perihelion (point of closest approach) of a planet in orbit around a stationary ring-shaped "star" of radius a and mass M. The planet orbits in the plane of the ring and its distance R, from the center of the ring satisfies $R \gg a$.



The situation in this problem is a toy model for the effects of an oblate Sun. Professor Dicke (and others) pointed out that an oblate Sun could be responsible for part of the excess precession of the perihelion of Mercury—an effect usually attributed entirely to general relativity.

a) What is the gravitational potential of the ring in the plane of the ring? Include terms to order $(a/R)^2$.

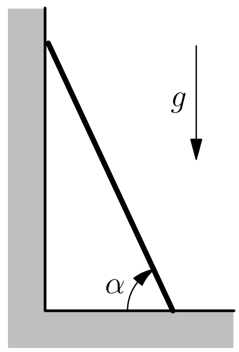
b) What is the angular velocity, ω_0, of a circular orbit of radius R, to order $(a/R)^2$?

c) If the planet is given a small radial perturbation, its new orbit will oscillate about the original circular orbit with angular frequency ω_r. Find an expression for the precession of the perihelion, $\Delta\phi = 2\pi(\omega_r - \omega_0)/\omega_0$.

M00M.2—Ladder on a Wall

Problem

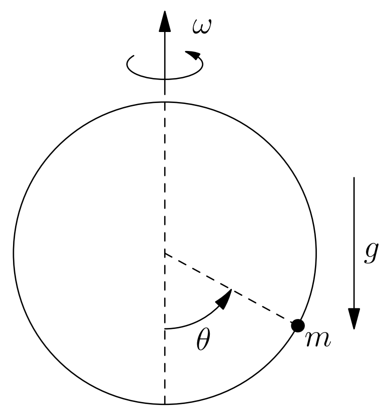
A uniform ladder leans against a frictionless vertical wall and rests on a frictionless horizontal floor. It is released from rest, with the ladder and the floor initially making an angle α. At some point, the ladder will separate from the wall. Determine the angle the ladder makes with the floor when this happens.



M00M.3—Bead on a Hoop

Problem

A circular hoop of radius a rotates about a vertical diameter with constant angular velocity ω. A small bead of mass m is constrained to slide without friction on the hoop. Consider the case when $\omega^2 = g/a$. The bead can undergo small oscillations around $\theta = 0$. These are not simple harmonic oscillations! Determine the period of these small oscillations. You may leave an unevaluated definite integral in your expression, but your solution should make it obvious how the period depends on the amplitude of oscillation.



M00E.1—Emitted Flux Density

Problem

A conductor is at temperature T in a vacuum. The goal of this problem is to deduce the flux density emitted from the conductor at angle θ from the normal to its surface. Recall Kirchhoff's law of heat radiation (as clarified by Planck):

$$\mathcal{E}_\nu = A_\nu K(\nu, T) = A_\nu \frac{h\nu^3/c^2}{e^{h\nu/kT} - 1},$$

where A_ν is the unitless absorption coefficient and $\mathcal{E}_\nu$ is the flux density (power per area per frequency) emitted from a body at temperature T. Specifically, one can write:

$$A_\nu = 1 - R_\nu = 1 - \left| \frac{E_r}{E_i} \right|^2 .$$

Also recall Fresnel's equations of reflection:

$$\left.\frac{E_r}{E_i}\right|_\perp = \frac{\sin(\theta_t - \theta_i)}{\sin(\theta_t + \theta_i)}, \quad \left.\frac{E_r}{E_i}\right|_\parallel = \frac{\tan(\theta_t - \theta_i)}{\tan(\theta_t + \theta_i)},$$

where i, r, and t label the incident, reflected, and transmitted waves, respectively, and $\perp$ and $\parallel$ refer to the plane of emission.

a) Begin by finding an approximate expression for the complex wave vector k_t as a function of frequency for the wave transmitted into a good conductor ($4\pi\sigma/\epsilon\omega \gg 1$, and $\mu \approx \mu_0$).

b) Find $\mathcal{E}_{\nu_\perp}$, the flux density emitted polarized perpendicular to the plane of emission.

c) Find $\mathcal{E}_{\nu_\parallel}$, the flux density emitted polarized parallel to the plane of emission.

d) Comment briefly on the polarization of the thermally emitted radiation in the grazing case ($\theta \to 90°$).

M00E.2—Particle Above a Conducting Plane

Problem

A particle of mass m and charge q is released from rest from a distance z_0 above an infinite grounded conducting plane. Neglect relativistic effects and gravity.

a) How long will it take for the particle to hit the plane? (Neglect radiation loss.) You may leave your answer in terms of a *dimensionless* integral.

b) What is the power radiated as a function of z?

Now consider the conducting plane to be replaced by a semi-infinite dielectric ϵ. (That is, for $z > 0$, there is a vacuum, and for $z < 0$, space is filled with the dielectric.)

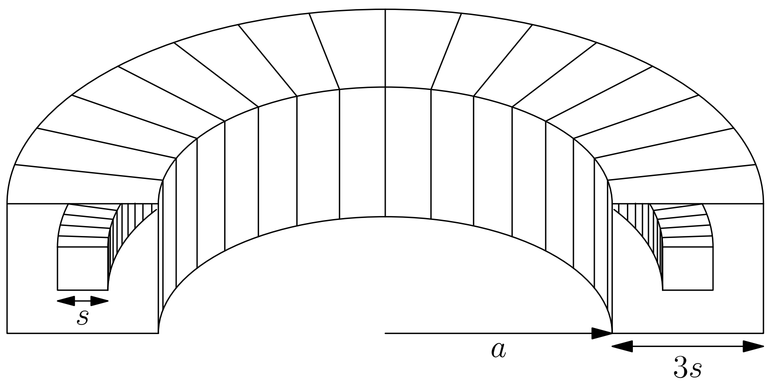
c) Calculate the force on the charge q when it is a distance z_0 above the plane.

Hint: an image solution exists where image charges are placed at either $+z_0$ or $-z_0$.

M00E.3—Toroidal Solenoids

Problem

The sketch shows two concentric toroidal solenoids. The outer one has N_1 turns of wire, each with square cross-section with side $3s$. The inner diameter of the outer solenoid is **a**, as shown in the figure. The inner solenoid has N_2 turns of wire, with square cross-section with side s. The resistance of the outer wire is negligible, but the resistance of the inner wire is $\mathcal{R}$.



a) If an AC voltage of amplitude V_0 and angular frequency ω is applied to the outer wire, what is the power dissipated in the system? You may leave your answer in terms of parameters given and the inductances of the system.

b) Find expressions for the relevant inductances.

M00Q.1—Harmonic Oscillator in a Magnetic Field

Problem

Consider a spin-$\frac{1}{2}$ particle constrained to move on a 1D line with a harmonic oscillator potential and a magnetic field so that the Hamiltonian is:

$$H = \frac{1}{2m}p^2 + \frac{1}{2}m\omega^2 x^2 + \omega S_z.$$

The first energy level is not degenerate but all the other levels are doubly-degenerate.

Now add a small magnetic field in the $\hat{x}$ direction with a magnitude proportional to x. The Hamiltonian is:

$$H = \frac{1}{2m}p^2 + \frac{1}{2}m\omega^2 x^2 + \omega S_z + \alpha x S_x.$$

Calculate the energy difference in the levels to lowest order.

M00Q.2—Hyperfine Structure

Problem

The hyperfine energy structure of a hydrogen atom in the $1s$ level in a constant magnetic field $\vec{B}$ may be represented by the Hamiltonian

$$H = 4\alpha \vec{s}_p \cdot \vec{s}_e + 2\beta \vec{s}_p \cdot \vec{B} + 2\gamma \vec{s}_e \cdot \vec{B}$$

where α, β, and γ are constants and the spin observables are $\vec{s}_p$ and $\vec{s}_e$.

Find the energy eigenvalues.

M00Q.3—Emission of Alpha Particles

Problem

A massive particle X with spin 2 decays into a spin 0 particle with no orbital angular momentum and with the simultaneous emission of two alpha particles, each of which is known to be in a *p*-wave. Given an ensemble of unpolarized X particles at rest, what is the probability distribution in the angle between the directions at which the two alpha particles are emitted?

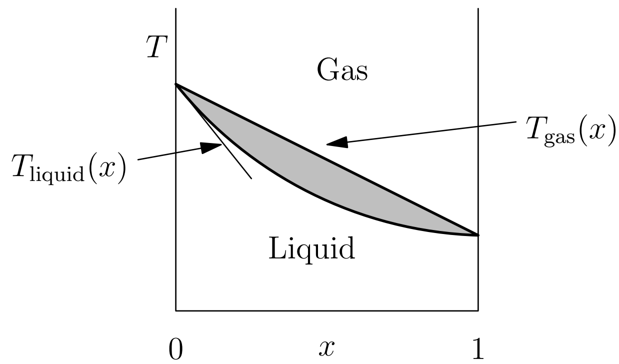
Note that you can simplify the problem by choosing an axis of quantization for the angular momenta, and then considering, in an ensemble of these objects, those cases in which the alpha particles both are discovered to be moving away from the decay at polar angle $\theta \sim \pi/2$ and at azimuthal angles ϕ_1 and ϕ_2.

M00T.1—Distillation

Problem

Sketched below is the phase diagram for a solution of two substances, A and B, at pressure $P = 1$ atm, where x is the mass fraction $x = \frac{M_A}{M_A + M_B}$. Assume that in the regime of interest, the boundaries of the two phase regime are sufficiently approximated by the linear functions:

$$T_{\text{gas}}(x) = T_0 - x\ ,$$
$$T_{\text{liquid}}(x) = T_0 - 3x\ .$$



The shaded region indicates phase coexistence.

A beaker has initially a mixture of liquids at the mass fraction $x_i = 0.2$. The liquid is brought to boil, and maintained at a boiling temperature, at atmospheric pressure.

a) Does the boiling increase or decrease the concentration of the A substance in the liquid?

b) What portion of the liquid needs to be boiled off in order to change the concentration of A (in the liquid remaining in the beaker) by a factor of 2?

M00T.2—Brownian Motion

Problem

A solid spherical particle of radius b and mass M is suspended in a fluid, and is seen, using an optical microscope, to undergo Brownian motion. You are asked to show that a measurement of the mean-square displacement $\langle[\vec{r}(t_1) - \vec{r}(t_2)]^2\rangle$ can be used to determine Boltzmann's constant.

Assume the densities of the solid and fluid are identical, so buoyancy can be ignored. The cause of the Brownian motion is a rapidly fluctuating force due to collisions with the molecules of the fluid. The force has mean zero, $\langle\vec{F}(t)\rangle = 0$, and two-time correlation of the form

$$\langle\vec{F}(t) \cdot \vec{F}(t')\rangle = C\delta(t - t').$$

The fluid has viscosity η and the system is isothermal at temperature T. The equation of motion of the particle is

$$M\frac{d^2\vec{r}}{dt^2} + 6\pi\eta b\frac{d\vec{r}}{dt} = \vec{F}(t).$$

a) Express the velocity at time t as an integral involving the past forces, $\{\vec{F}(t')\}_{t'<t}$.

b) Find the coefficient C of the force-correlation at temperature T, as a function of T and the other constants mentioned above.

c) Describe the rate of growth of the mean square displacement $\langle|\vec{r}(t_1) - \vec{r}(t_2)|^2\rangle$, and explain how its measurement can be used to determine Boltzmann's constant k_B.

M00T.3—Defects in a Crystal

Problem

In a crystal lattice, a defect is created when an atom hops from a lattice site to an interstitial site. The ground state is a configuration with no defects. However, when the lattice is in *equilibrium* at a finite temperature T, defects appear spontaneously.

Consider the case where the number, N, of atoms is equal to the number of lattice sites and the number of possible interstitial sites is N_i. Consider the thermodynamic limit where $N,\ N_i \to \infty$ at constant $N_i/N = \rho$. The energy required to create a defect is ε. Denoting by K the number of defects, let $n = K/N$ be their density.

a) Find an expression for the *free energy* per particle at temperature T.

b) Calculate the density of defects, $n(T)$.

c) Sketch the curve of $n(T)$ versus T and describe the behavior of $n(T)$ as $T \to 0$.

d) Calculate the entropy S and the heat capacity C *due to the defects* at temperature T.