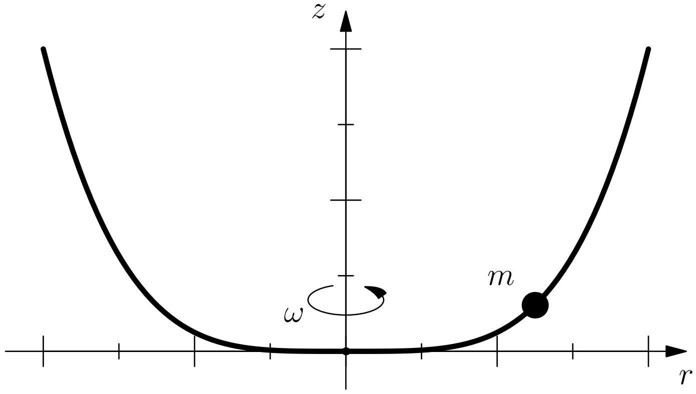
# J04M.1—Bead on a Wire (J06M.3)

## Problem

A bead of mass $m$ slides without friction on a wire whose shape is

$$z(r) = a\left(\frac{r}{a}\right)^4$$

The wire rotates about the z axis with constant angular velocity $\omega$. Earth's gravity causes acceleration $g$ in the negative $z$ direction.




a) Find the equation of motion for the bead in terms of coordinate $r$.

b) Find the equilibrium points. Say whether each is stable.

c) For the stable equilibria, find the frequency of small oscillations about equilibrium.

# J04M.2—Power Law Central Force (J06M.2)

## Problem

A particle of mass $m$ moves in an attractive, power-law central force, $F \propto r^{-n}$. Define constant $k$ so that the potential energy is

$$U(r) = -\frac{k}{r^{n-1}}.$$

The particle is set moving from a position near the force center with such initial conditions that its orbit is a circle of radius $a$ which passes through the origin $r(\theta) = 2a\cos\theta$. The existence of this orbit imposes conditions on the force law, the energy, and the angular momentum.

a) Find the force power law $n$.

b) Find the angular momentum of the particle in terms of the constants $k, a$ and $m$.

c) Show that the total energy, $E$ vanishes for this orbit.

d) How long does it take for the particle to complete a full trip from the force center and back to it?

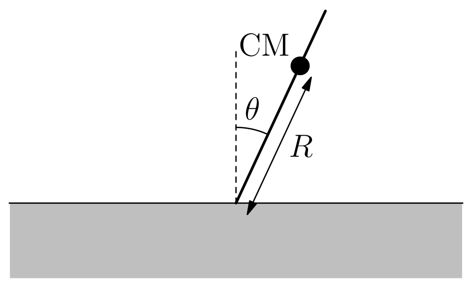
e) Remove the orbiting mass $m$ to infinity and give it an initial velocity $v_\infty$. Find the total cross section for the force described above to "capture" the particle (i.e. for the particle's trajectory to pass through the force center at $r = 0$).

# J04M.3—Falling Stick

## Problem

A thin stick with some arbitrary linear mass density $\mu(x)$ along it is initially at rest. It has one end on a table and makes an angle $\theta_0$ with the vertical. The stick-table contact point has an infinite coefficient of friction.

Let $m$ be the total mass of the stick, $R$ be the distance from the contact point to the center of mass, $I_{CM}$ be the moment about the center of mass, and $g$ be the acceleration due to gravity.




a) The stick is released from rest and allowed to fall to the table. Finad the condition that the end of the stick initially in contact with the table *does* rise from the table as the stick falls. Express the condition in terms of $\theta_0, m, g, R$, and $I_{CM}$.

b) Now consider a specific mass distribution. Let the mass be uniformly distributed along the length. For what range of initial angles $\theta_0$ will the stick eventually lift off the table?

c) Consider a different mass distribution: the mass is concentrated in two points of equal mass, one at either end of the stick. Now for what range of initial angles $\theta_0$ will the stick eventually lift off the table?

# J04E.1—Rotating Cylindrical Capacitor

## Problem

Consider a long cylindrical capacitor, which consists of two metallic concentric cylinders of length $l$. The radii of the cylinders are $a$ (outer) and $b$ (inner). $l \gg a, b$. The axes of the two cylinders coincide with the $z$-axis. As an added bonus the cylinders are free to rotate about the $z$-axis independently from each other and without friction. The voltage between the two conductors is $U$. Originally, at time $t = 0$ there is also a magnetic field, $\boldsymbol{B}_0 = B_0 \boldsymbol{e}_z$ in the $z$-direction.

a) Determine the charge on each of the cylinders at $t = 0$ and the electric field, $\boldsymbol{E}(\rho)$, (magnitude and direction), in the volume between the cylinders as functions of the distance $\rho$ from the axis. As in all parts of this exam, either MKSA or Gaussian units may be employed.

b) The magnetic field is slowly reduced, remaining parallel to the $z$-axis, until it vanishes at some moment of time, $t_0$. This causes the two cylinders to start rotating. Use Faraday's law to determine the angular momenta, $\boldsymbol{L}_i$ and $\boldsymbol{L}_o$ of each cylinder after $t_0$. YOu may ignore the magnetic field produced by the rotation of the cylinders. Also ignore any fringing of the fields at the ends of the cylinders.

c) Recall that the electromagnetic fields contain a momentum density, $\boldsymbol{S} = \boldsymbol{E} \times \boldsymbol{B}/4\pi c$ (Gaussian units). Taking this fact into account, evaluate the angular momentum (magnitude and direction) contained in the electromagnetic field in the initial configuration. Compare it with the total angular momentum of the capacitor in the final configuration.

d) How will the result for the angular momentum of the two cylinders change (increase, decrease or remain the same) if the magnetic field of the rotating cylinders were taken into account?

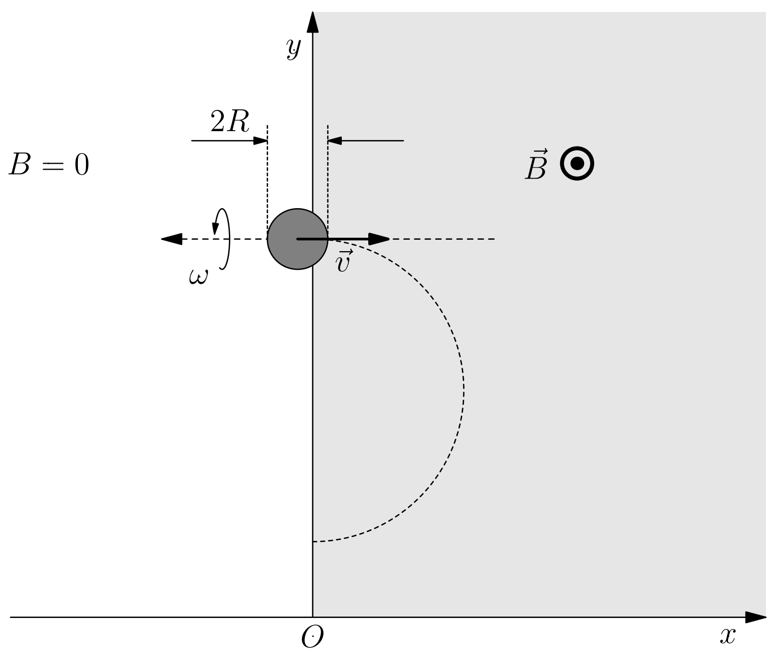
# J04E.2—Mass Spectrometer

## Problem

Let the magnetic field, $\boldsymbol{B}$, have the configuration which is used in mass spectrometers: $\boldsymbol{B} = 0$ for $x < 0$, while for $x > 0$ it is uniform, $\boldsymbol{B} = B_0\hat{\boldsymbol{z}}$. A spherical ball with radius $R$, total mass $M$ and total charge $Q$ approaches the plane $x = 0$ from the left and enters the magnetic field region $x > 0$ with center of mass velocity $v$ in the $x$-direction. In addition, the ball rotates with angular velocity $\omega$ about the $x$-axis. The density of the ball is distributed uniformly throughout its volume.

The ball enters the region $x > 0$, spends there some time $t$ and then leaves the region.

Both the speed $v$ and angular velocity $\omega$ are small, so that relativistic effects can be neglected. The ball is so small that you can ignore all the rotational dynamics during the time $2R|v|$ when it traverses the boundary at $x = 0$.




a) Determine the time $t$ the ball was subject to the magnetic field.

b) Assume that the charge of the ball $Q$ is uniformly distributed throughout its volume, as the mass is, and evaluate the initial angular momentum and magnetic moment of the ball.

c) Determine the direction of the angular momentum after the ball comes back to the region where $\boldsymbol{B} = 0$.

d) Suppose instead that the charge $Q$ were uniformly distributed over the *surface* of the ball, while the mass remains uniform throughout the *volume*. What is the initial magnetic moment? How is the final magnetic moment directed?

# J04E.3—Thomson Scattering

## Problem

A particle of mass $m$ and charge $q$ moves at a constant, nonrelativistic speed $|\boldsymbol{u}|$ in a circle of radius $a$. The plane of the orbit coincides with the $x$-$y$ plane This motion is caused by a plane circularly polarized electromagnetic wave, which propagates in the $z$-direction. At any moment of time the magnetic field of the electromagnetic wave is parallel to the velocity $\boldsymbol{u}$.

Since the acceleration of the particle $\dot{\boldsymbol{u}}$ differs from zero, the particle emits radiation. For non-relativistic particles the radiation electric field $\boldsymbol{E}_{rad}$ at point $\boldsymbol{R} = \boldsymbol{n}r$, $r \gg a$ can be computed from

$$\boldsymbol{E}_{rad} = \frac{q}{rc^2}\boldsymbol{n} \times (\boldsymbol{n} \times \dot{\boldsymbol{u}}),$$

where $\boldsymbol{n}$ is the direction of the emission vector (The origin is at the center of the orbit).

a) Detemine the power emitted per unit solid angle in the direction at angle $\theta$ relative to the $z$-axis.

b) What is the spectrum of the emitted radiation?

c) By relating the emitted power to incident flux of the plane electromagnetic wave find the total cross section for Thomson scattering of unpolarized radiation and express it in terms of $m$ and $q$.

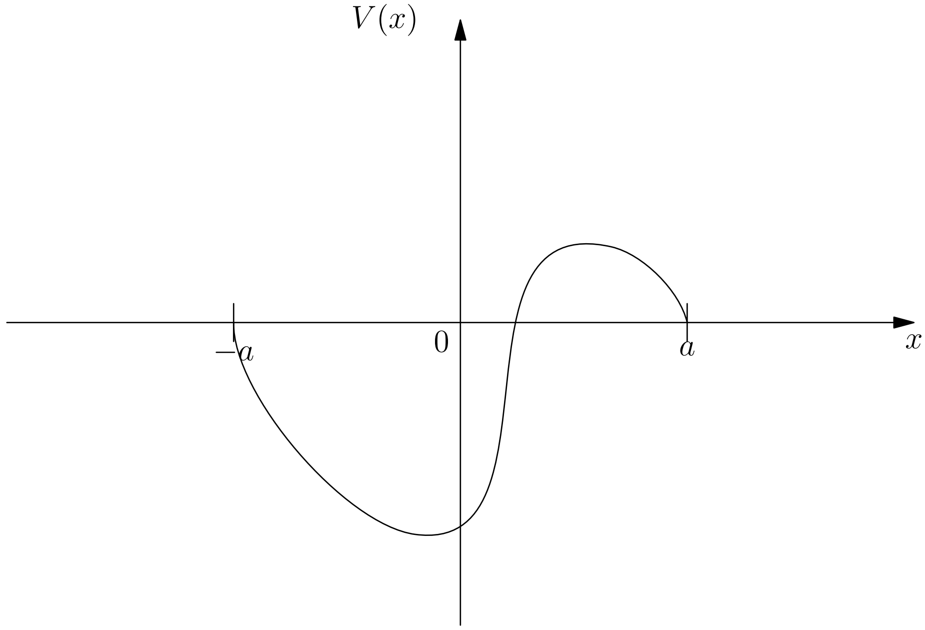
# J04Q.1—Arbitrary Potential

## Problem

Consider a particle in *one* dimension moving under the influence of a potential $\lambda V(x)$, with $V(x) \to 0$ as $|x| \to \infty$. The hamiltonian is

$$H = \frac{p^2}{2m} + \lambda V(x).$$

For simplicity, assume that the potential is non-zero only in the region $|x| < a$.




a) Use the variational principle to show that as long as $\int V(x)dx < 0$ this Hamiltonian has a bound state for *arbitrarily weak* but positive coupling constant $\lambda$.

b) Give an upper bound for the energy of this bound state for $\lambda \ll 1$.

c) Would the same approach work in three dimensions? Explain why.

# J04Q.2—Boosted Wave Function

## Problem

Two observers in different inertial frames will need different wave functions to describe the same physical system. To make things simple we will consider how it works in one dimension: The first observer uses coordinates $(x, t)$ and a wave function $\psi(x, t)$ while the second uses $(x', t)$ and $\hat{\psi}(x', t)$ with, of course, $x' = x - vt$, $v$ a constant velocity. The wave functions for the two observers are said to be related as follows:

$$\hat{\psi}(x', t) = \psi(x, t) \exp\left(-\frac{i}{\hbar}\left[mvx - \frac{m}{2}v^2 t\right]\right).$$

Despite its innocuous look (it's just a phase!) this transformation has interesting effects!

a) Let's verify that it makes sense. Suppose $\psi(x, t)$ is the wave function for a free particle of momentum $p$. Show that $\hat{\psi}(x', t)$ is the wave function of a free particle with a different momentum. What is its momentum?

b) Now let's put this to work. Suppose we have a harmonic oscillator in its ground state for $t < 0$; its wave function is

$$\psi(x, t) = N \exp\left(-\frac{m\omega}{2\hbar}x^2 - \frac{i}{2}\omega t\right),$$

where $N$ is a normalization constant. Suppose that at $t = 0$ the potential suddenly starts to move at velocity $v$. Because the wave function does not change immediately, it is no longer the ground state wave function of the moving harmonic oscillator. What is the probability of finding the system in the moving ground state at a later time $t > 0$?

# J04Q.3—Spin in a Magnetic Field

## Problem

A spin 1/2 particle of magnetic moment $\mu$ has spin up along the $z$ direction. At time $t = 0$ the magnetic field $\vec{B} = B\hat{y}$ is turned on.

a) Calculate the expectation value of spin $\vec{S}$ as a function of time. Compare the result with the classical answer.

b) At $t = T$ what is the probability that the spin is down?

c) At $t = T/2$ another experimentalist measures the $z$-component of the spin but does not tell you the result. What is the probability that your subsequent measurement at $t = T$ will find the spin to be down?

# J04T.1—Non-interacting Spins

## Problem

Consider $N$ non-interacting quantized spins in a magnetic field $\vec{B} = B\hat{z}$. The energy of the spins is $-BM_z$, where

$$M_z \equiv \mu \sum_{i=1}^{N} S_z^{(i)}$$

is the total magnetization. For each spin, $S_z^{(i)}$ takes only $2S+1$ values $-S, -S+1, \ldots, S-1, S$. Given the temperature of the system $T$:

a) Calculate the Gibbs partition function $Z(T, B)$;

b) Calculate the Gibbs free energy $G(T, B)$ and evaluate its asymptotic behavior at weak ($\mu BS \ll k_B T$) and strong ($\mu B \gg k_B T$) magnetic field;

c) Calculate the zero-field magnetic susceptibility

$$\chi \equiv \left( \frac{\partial M_z}{\partial B} \right)_{B=0}$$

d) Calculate the magnetic susceptibility at strong fields $\mu B \gg k_B T$.

# J04T.2—Fermionic Gas

## Problem

Consider a gas of $N$ nonrelativistic fermions with spin 1/2 and mass $m$ initially at zero temperature and confined in a volume $V_0$.

a) Express the kinetic energy of the gas in terms of $N$ and $V_0$.

b) What is the pressure of the gas? You can assume here that the gas is ideal.

c) Now the gas is allowed to expand to the volume $V_1 \gg V_0$ without any energy exchange with the outside world. Calculate the temperature of the gas after it will reach equilibrium due to weak interactions between the fermions.

d) What is the pressure of the gas in the final state?

# J04T.3—DNA as a Polymer Chain

## Problem

The statistical mechanics of biological polymers is important for understanding such things as the properties of DNA or protein folding. The simplest "ideal gas"-like description of a polymer treats the dominant free energy terms as coming from the *conformational entropy*: typical polymer configurations can be modeled as a random walk of $N$ steps. The height of each step $b$ represents the lengthscale of a flexible microscopic polymer "link". This leads to an "ideal polymer" free energy $F_0$ as a function of the distance between the two free ends $|\vec{R}|$

$$F_0(|\vec{R}|, N) = -TS_{\text{config}} = \text{const} + k_B T \frac{3R^2}{2Nb^2}.$$

a) Use this expression to find the *mean square* end-to-end distance $\langle R^2 \rangle$. How does it depend on $N$?

b) What is the *most probable* end-to-end distance?

As in the case of the "ideal gas", the "ideal polymer" picture neglects the interactions between different parts of the polymer. The polymer may be viewed as a ball with radius of order $R$. The principle "two body" interaction is the excluded volume effect: two links cannot occupy the same point in space. Take $v$ as the "excluded volume" of the region around a link into which a second link of the polymer cannot enter. One can develop a low-density expansion for the interaction energy in powers of $vR^3$.

You are asked to investigate the leading (two-link) interaction correction to the free energy in the low-density limit.

$$F(|\vec{R}|, N) = -TS_{\text{config}}(|\vec{R}|, N) + vU(|\vec{R}|, N)$$

c) Use dimensional analysis and low density arguments to determine $U$ (up to a constant factor) in the limit $v \to 0$.

d) Use your answer to (c) to estimate the equilibrium (most probable) end-to-end distance of the non-ideal polymer as a function of $N, b$, and $v$. (Your results should reveal a different predicted $N$-dependence as compared to the "ideal polymer"!)