J03M.1—Scattering from an Attractive Potential

Problem

This problem is about scattering by an attractive potential.

a) Consider a particle with energy E and $z < 0$ approaching the $z = 0$ plane at an angle θ_1 to the z-axis. Find the angle θ_2 that it makes to the z axis after passing through the $z = 0$ plane if $V = 0$ for $z < 0$ and $V = -V_0$ (constant) for $z > 0$.

b) Apply your result to a uniform beam of particles scattered by the attractive potential

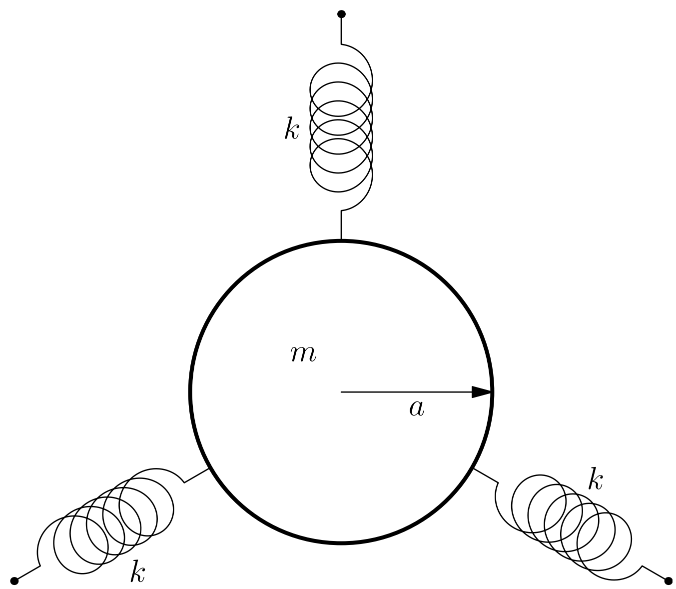
$$V(r) = -V_0 \quad r < a, \qquad V(r) = 0 \quad r > a$$

Determine the differential cross section. (Recall that the definition of the differential cross section is $\frac{d\sigma}{d\Omega} = \frac{b}{\sin\theta}\frac{db}{d\theta}$, where b is the impact parameter and θ the scattering angle.)

J03M.2—Disk with Three Springs

Problem

A uniform disk of mass m and radius a rests on a horizontal frictionless surface. It is symmetrically attached to three identical, ideal, massless springs whose other ends are attached to the three vertices of an equilateral triangle.



At equilibrium, the length l of the springs is greater than the relaxed length l_0. The disk remains in the initial horizontal plane but is otherwise free to move. (The diagram shows the view looking down on the plane.) What are the frequencies of the normal modes of small oscillations? What do the modes look like? Hint: You might identify the normal modes from the symmetries before calculating the frequencies.

J03M.3—Orbits in a Central Potential

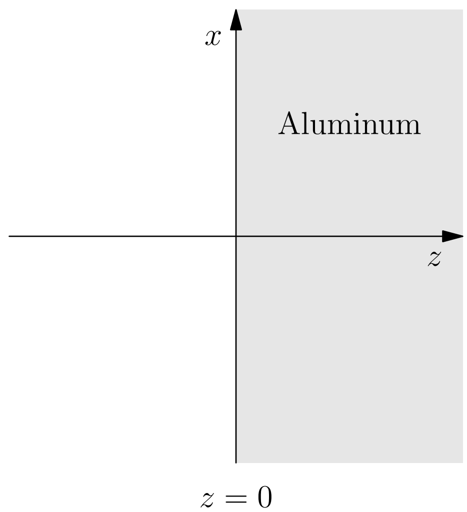
Problem

Consider a particle of mass m moving in a fixed central potential with angular momentum l. The potential is $V(r) = -C/2r^2$ where r is the distance from the center and $C > 0$ is a constant. Note that the potential leads to an attractive central force, $F(r) = -C/r^3$. There are various kinds of orbits in this potential depending on the particle's angular momentum and energy. Determine those values of the parameters which separate the different classes of orbits and give an example, including a sketch, of each class of orbit. By orbit is meant r as a function of azimuthal angle, ϕ, in the plane of the orbit.

J03E.1—Waves in Aluminum

Problem

A plane wave at 90 GHz is normally incident on aluminum, a very good conductor, that fills the space with $z > 0$ as shown in the figure. Most, but not all, of the field is reflected from the surface. Aluminum has a magnetic permeability equal to that of free space and a conductivity of $\sigma = 3.5 \times 10^{17}\ \mathrm{s}^{-1}$ or $3.5 \times 10^{7}\ \Omega^{-1}\mathrm{m}^{-1}$.



a) Assume the wave inside the aluminum has the form

$$\vec{E} = E_0 \exp(ikx - i\omega t)\hat{e}_x.$$

What is the dispersion relation, $k(\omega)$, in the aluminum?

b) What fraction of the incident power is reflected?

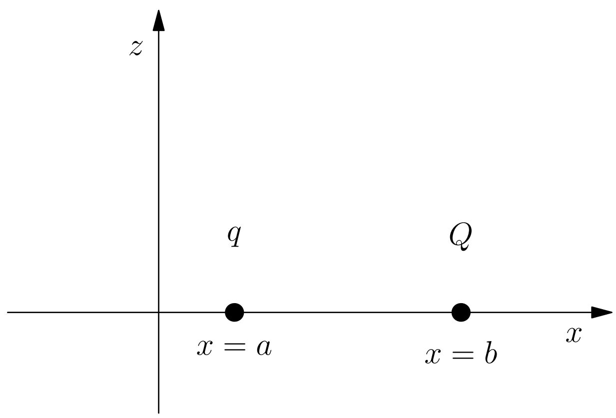
c) What is the numerical value of the normal emissivity ϵ? Recall that ϵ = (the power emitted)/(the power emitted by a perfect radiator at the same temperature).

J03E.2—Image Charges

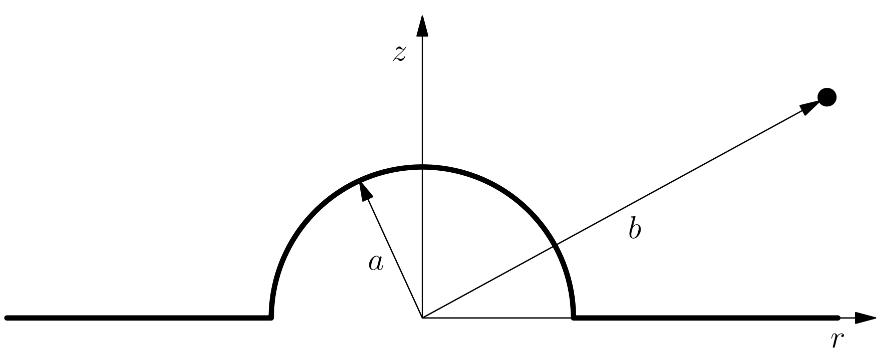
Problem

This problem contains three questions on electrostatics.

a) A charge Q is at $x = b$ and a second charge $q = -Q\sqrt{\frac{a}{b}}$ is at $x = a$. Show that the equipotential surface corresponding to $V = 0$ is described by a sphere with its center at the origin. Determine the radius R of this sphere.



b) FInd the electric potential in cylindrical coordinates $\phi(r, \theta, z)$ when a charge q is located at $(r_0, z_0 > 0)$ and there is a grounded conducting plane at $z = 0$ that has a conducting hemispherical boss of radius $R < b = \sqrt{r_0^2 + z_0^2}$ whose center is at the origin. A side view of the boss and conducting plane is shown in the picture below.

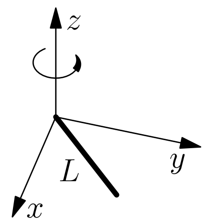


c) What is the electrostatic force on the charge q in part b) for the case that $r_0 = 0$?

J03E.3—Rotating Charged Rod

Problem

A rod of length L and negligible cross sectional area carries a total charge Q uniformly distributed along its length. It is rotated slowly with $\omega \ll c/L$ in the x-y plane as shown.



a) What are the electric dipole moment and rate at which electric dipole energy is radiated?

b) What are the magnetic dipole moment and rate at which magnetic dipole energy is radiated?

J03Q.1—Central Potential Scattering

Problem

Consider the scattering of quantum-mechanical particles by a spherical square-well in three dimensions given by a radial potential

$$V(r) = \begin{cases} V_0 & \text{for} \quad r < a, \\ 0 & \text{otherwise} \end{cases}$$

having a constant depth V_0 within a radius $a > 0$ of the origin. Assume that the particles have an extremely low low energy $E > 0$, that is, $a\sqrt{2mE} \ll \hbar$. In this case only partial waves of angular momentum $L = 0$ suffer appreciable scattering.

a) Calculate the total cross section for the case of an attractive potential with depth $V_0 < 0$.

b) Starting from the answer you derived, consider now the case of scattering from a hard sphere, by taking the potential to be repulsive ($V_0 > 0$) in the limit $V_0/E \to \infty$. Show that the answer is $4\pi a^2$ (four times bigger than the classical result).

J03Q.2—Hydrogen Atom Transitions

Problem

An isolated hydrogen atom in the $2s$ level has a very long lifetime for radioactive decay because selection rules pretty much force it to decay by two-photon emission. In realistic situations, the atom suffers collisions that push the $2s$ level into the $2p$ levels, from which it rapidly decays by standard electric dipole emission.

In plasmas, the collisions are with ions that briefly subject the hydrogen atom to an electric field. Let us study what happens when an ion of charge Q, moving at constant velocity v passes by the H atom, making a closest distance of approach b. The electron in the atom sees a time-dependent potential

$$V_1(\vec{x}, t) = \frac{Qe}{|\vec{b} + \vec{v}t - \vec{x}|} \qquad \vec{b} \cdot \vec{v} = 0.$$

Because the ion passes far from the atom, you can treat x as small and expand in powers of x. Keep the term of first order in x and treat it as the perturbing potential that induces transitions between the degenerate states of the $n = 2$ levels of hydrogen (the zeroth-order term doesn't depend on x and causes no transitions).

Use first-order time-dependent perturbation theory to find the transition amplitude for an atom originally in the $2s$ level to wind up in one of the $2p$ levels.

You will need some hydrogen wave functions:

$$\begin{aligned}
\phi_{2s} &= \frac{1}{2\sqrt{2\pi a_B^3}}(1 - r/2a_B)e^{-r/2a_b} \\
\phi_{2p,0} &= \frac{z}{4\sqrt{2\pi a_B^5}}e^{-r/2a_b} \\
\phi_{2p,\pm 1} &= \frac{x \pm iy}{8\sqrt{2\pi a_B^5}}e^{-r/2a_b}
\end{aligned}$$

J03Q.3—Nuclear Alpha Decay

Problem

An unpolarized nucleus of spin $S = 2$ decays into a nucleus of spin 0, plus two alpha particles, both having spin 0 and orbital angular momentum $L = 1$. We would like to predict the probability distribution of the angle between the directions of motion of the outgoing alphas. (Assuming the original nucleus is unpolarized, there are no other meaningful angles in the problem.)

a) As a first step, use the techniques of angular momentum addition to construct states of total angular momentum 2 out of two particles of orbital angular momentum 1. Put otherwise, find the normalized linear combinations of $Y_{1m_1}(\theta_1, \varphi_1)Y_{1m_2}(\theta_2, \varphi_2)$ that provide a basis of the total angular momentum 2 representation.

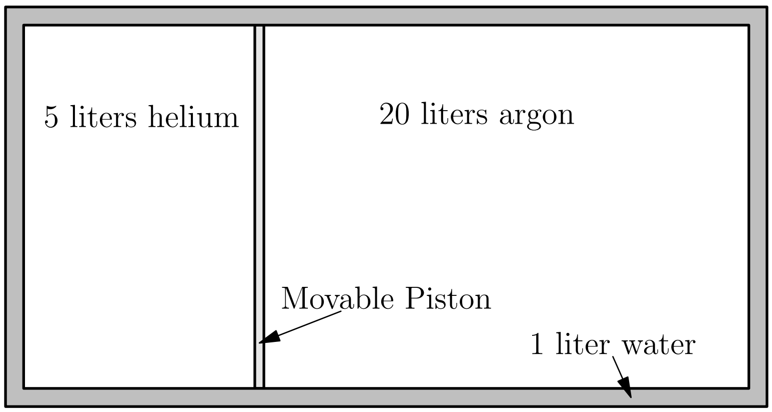
b) Next, compute the probability density $p(\theta_1, \varphi_1; \theta_2, \varphi_2)$ for the joint angular distribution of both alphas when both $\theta_1 = \theta_2 = \pi/2$, so both alphas lie in the plane perpendicular to the S_z-quantization axis. Recall that the original $S = 2$ nucleus is unpolarized (*i.e.* has equal *probability* of being in the 5 different S_z substates).

c) The density obtained in the preceding part only depends on the angle $\omega = \varphi_1 - \varphi_2$ between the two particles. Use the fact that even for general $\theta_{1,2}$ and $\varphi_{1,2}$, the density p will only depend on the angle ω between the directions of the two alphas to compute the probability distribution of ω.

J03T.1—Container of Gases

Problem

A container is filled with 5 liters of helium gas and 20 liters of argon gas separated by a movable thin piston. The walls of the container are hollow and are filled with one liter of water. The heat capacity of the piston and the container walls are negligible. The piston is initially in its rest position, and the water and the gas are in thermal equilibrium at a temperature of 32°C.



One moves the piston slowly, while maintaining thermal equilibrium, to the point where the pressure of the argon gas is twice its original value. Then, suddenly, the piston is released. The piston returns to its rest position and thermal equilibrium is restored. After the entire process the temperature of the water has increased by exactly 1.0° C. The heat capacity of water is 4.184 kJ/kg·K, and the gas constant is $R = 8.314$ J/mole·K.

a) How many moles of helium and argon are inside the container?

Next, the water is removed from the walls and the same process is repeated but now without the water.

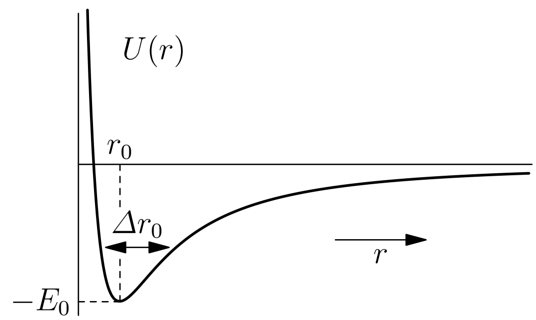
b) What is the pressure of the gas before and after this second process? Express your answer in units of 1 atmosphere = 1.01×10^5 N/m^2.

c) Determine the change in entropy during the first process and the second process. Is there a difference? Explain.

J03T.2—Krypton Molecule Formation

Problem

Krypton atoms are rather heavy and reasonably polarizable. The potential between two krypton atoms is shown in the figure



In the limit in which the mass of the Kr atom is very large, there will at low energies and temperatures be an equilibrium of the form

$$\mathrm{Kr} + \mathrm{Kr} \leftrightarrow \mathrm{Kr}_2$$

a) The classical partition function of two krypton atoms inside a volume V can be written as

$$Z_2 = \left(1 + \frac{K}{V}\right) Z_2^{\mathrm{id}}$$

where Z_2^{id} is the partition sum of two free atoms. How is the constant K related to the probability that the two atoms form a molecule? Find an approximate expression for K in terms of the reaction energy E_0, the size r_0 of the molecule, and the width Δr_0 of the potential.

b) Show that the partition function Z_N for N krypton atoms inside a volume V can similarly be written as a sum of contributions coming from M Kr_2 molecules and $N - 2m$ unbound free Kr atoms given by

$$Z_{N,M} = d(M,N) \left(\frac{K}{V}\right)^M Z_N^{id}$$

where Z_N^{id} is the ideal gas partition sum, $d(M,N)$ is the number of ways M molecules can be formed out of N atoms, and K is the same quantity found in part a). Determine the combinatorial factor $d(M,N)$.

c) Derive the equilibrium condition

$$c_{Kr_2} = K[c_{\mathrm{Kr}}]^2$$

where c_{Kr_2} is the concentration of the Kr_2 molecules and c_{Kr} the concentration of the unbound Kr atoms. You may use Stirling's formula $N! \sim \left(\frac{N}{e}\right)^N$.

J03T.3—Spin Gas in a Magnetic Field

Problem

Consider an ideal Boltzmann gas of N spin $\frac{1}{2}$ particles in a thermally isolated container at initial temperature T_i in a strong magnetic field H.

a) Compute the free energy and the entropy of the gas.

b) The magnetic field is slowly reduced to $H = 0$. Compute the final temperature of the gas. Express your answer in terms of the variable $x = \mu_B H / kT_i$.

c) Show that

$$T_i > T_f > 2^{-\frac{2}{3}} T_i.$$